The Various Operations of Fuzzy Approximations

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Abstract

We investigate the various operations of lower and upper approximations on a stsc quantale lattice \( L \).

Key Words : stsc-quantales, information spaces, lower and upper approximations, fuzzy relations.

1. Introduction and preliminaries

Rough set theory was introduced by Pawlak [14-16] with an equivalence relation. From both theoretical and practical viewpoints, the equivalence relation is a very strong condition that may limit applications of rough sets. Various extensions were developed from an equivalence relation to a fuzzy relation, covering, a neighborhood system in recent years [1-4,10,11]. Quantales have arisen in an analysis of the semantics of linear logic systems developed by Girard [5], which supports part of foundation of theoretic computer science. Höhle [7-9] developed the algebraic structures and many valued topologies in a sense of quantales and cqm-lattices. Bělohlávek [1-3] investigate the properties of fuzzy relations and similarities on a residual lattice.

In this paper, we study the various approximations as a generalization as fuzzy rough set in [4]. Moreover, we investigate the properties of various approximations with fuzzy relations and the relationship among them on a stsc quantale lattice \( L \).

Definition 1.1. [10-13] A triple \( (L, \leq, \oplus) \) is called a strictly two-sided, commutative quantale (stsc-quantale, for short) if it satisfies the following conditions:

1. \( L = (L, \leq, \lor, \land, 1, 0) \) is a completely distributive lattice where 1 is the universal upper bound and 0 denotes the universal lower bound;
2. \( (L, \oplus) \) is a commutative semigroup;
3. \( a = a \oplus 1 \), for each \( a \in L \);
4. \( \oplus \) is distributive over arbitrary joins, i.e.
   \[
   \bigvee_{i \in \Gamma} a_i \oplus b = \bigvee_{i \in \Gamma} (a_i \oplus b).
   \]

Remark 1.2. [10-13](1) A completely distributive lattice is a stsc-quantale. In particular, the unit interval \([0, 1], \leq, \lor, \land, 0, 1\) is a stsc-quantale.

   (2) The unit interval with a left-continuous t-norm \( t \), \([0, 1], \leq, t\), is a stsc-quantale.

   (3) Let \((L, \leq, \oplus)\) be a stsc-quantale. For each \( x, y \in L \), we define
   \[
   x \rightarrow y = \bigvee\{z \in L \mid x \oplus z \leq y\}.
   \]

Then it satisfies Galois correspondence, that is,
\[(x \circ y) \leq z \iff x \leq (y \rightarrow z).

Lemma 1.3. [10-13,17] Let \((L, \leq, \circ)\) be a stsc-quantale with a strong negation \( x^* \equiv x \rightarrow 0 \). We define \( x \circ y = (x^* \circ y^*)^* \) for all \( x, y \in L \). Let \( x, y, z, x_i, y_i \in L \) for all \( i \in \Gamma \), we have the following properties.

1. If \( y \leq z, (x \circ y) \leq (x \circ z), x \rightarrow y \leq x \rightarrow z \) and \( z \rightarrow x \leq y \rightarrow x \).
2. \( x \circ y \leq x \land y \leq x \lor y \).
3. \( x \rightarrow (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma} (x \rightarrow y_i) \).
4. \( (\bigvee_{i \in \Gamma} x_i) \rightarrow y = \bigvee_{i \in \Gamma} (x_i \rightarrow y) \).
5. \( x \rightarrow (\bigvee_{i \in \Gamma} y_i) \geq \bigvee_{i \in \Gamma} (x \rightarrow y_i) \).
6. \( (\bigwedge_{i \in \Gamma} x_i) \rightarrow y \geq \bigvee_{i \in \Gamma} (x_i \rightarrow y) \).
7. \( (x \circ y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) \).
8. \( x \circ (x \rightarrow y) \leq y \) and \( x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z) \).
9. \( y \circ z \leq x \rightarrow (x \circ y \circ z) \) and \( x \circ (x \circ y \circ z) \leq y \rightarrow z \).
10. \( (x \circ y)^* = x \rightarrow y^* \).
11. \( (x \rightarrow y) \circ (y \rightarrow z) \leq x \rightarrow z \).
12. \( x \rightarrow y = 1 \text{ iff } x \leq y \).
13. \( x \rightarrow y = y^* \rightarrow x^* \) and \( x \rightarrow y = (x \circ y^*)^* = x^* \oplus y \).

In this paper, we assume \((L, \leq, \oplus)\) is a stsc-quantale with a strong negation.

Definition 1.4. [4,10,11] Let \( U \) be a set of objects and \( V \) a set of attributes. A map \( R : U \times V \rightarrow L \) is called a fuzzy relation. The triple \((U, V, R)\) is called an information space.
Definition 1.5. [4,10,11] Let $U$ be a set. A function $R: U \times U \to L$ is called:

(R1) reflexive if $R(x,x) = 1$ for all $x \in U$,

(R2) symmetric if $R(x,y) = R(y,x)$ for all $x, y \in U$,

(R3) transitive if $R(x,y) \circ R(y,z) \leq R(x,z)$, for all $x, y, z \in U$.

If $R$ satisfies (R1) and (R2), then $R$ is an $\odot$-quasi-equivalence relation. If an $\odot$-quasi-equivalence relation $R$ satisfies (R2), then $R$ is an $\odot$-equivalence relation.

Theorem 1.6. [11] Let $R_1 \in L^U \times V$ and $R_2 \in L^V \times W$ be fuzzy relations. The compositions of $R_1$ and $R_2$ are defined as

$$R_1 \circ R_2(x,z) = \bigvee_{y \in V} R_1(x,y) \odot R_2(y,z)$$

$$(R_1 \Rightarrow R_2)(x,z) = \bigwedge_{y \in V} (R_1(x,y) \rightarrow R_2(y,z))$$

$$(R_1 \Leftarrow R_2)(x,z) = \bigwedge_{y \in V} (R_2(y,z) \rightarrow R_1(x,y))$$

where $R_1 \Rightarrow R_2 = \bigwedge_{y \in V} (R_1(x,y) \rightarrow R_2(y,z))$.

Then we have the following properties.

(1) $(R_1 \circ R_2)^* = R_2^* \circ R_1^*$.

(2) $(R_1 \circ R_2)^* = R_1 \Rightarrow R_2^* = R_2 \Rightarrow R_1 = R_1^* \odot R_2^*$.

(3) $(R_1 \Rightarrow R_2)^* = R_2^* \Leftarrow R_1^* = (R_2^*)^* \Rightarrow (R_1^*)^*$.

(4) $(R_1 \Leftarrow R_2)^* = R_2^* \Rightarrow R_1^*$.

(5) $(R_1 \Leftarrow R_2)^* = R_2^* \Rightarrow R_1^*$.

Theorem 1.7. [11] Let $R \in L^U \times U$ be a fuzzy relation. We have the following properties.

(1) If $R$ is reflexive, then $R \odot R$ is reflexive, $R \leq (R \odot R)$,

$$(R \Rightarrow R) \leq R, (R^* \Rightarrow R) \leq R, (R \Leftarrow R) \leq R$$

and $(R \Leftarrow R)^* \leq R$.

(2) $(R \circ R)^* = (R^* \circ R^*)^*$. If $R^*$ is reflexive, then $R \odot R \leq R$.

(3) $R$ is symmetric iff $(R \Rightarrow R)$ is reflexive iff $(R \Leftarrow R)$ is reflexive.

(4) If $R$ is symmetric, then $R \odot R$ is symmetric, $(R \Leftarrow R)^* = R \Rightarrow R, (R \Rightarrow R)^* = R^* \Rightarrow R^*$ and $R \Leftarrow R$ is symmetric and reflexive.

(5) $R^* \circ R \leq R$ iff $R \leq (R \Rightarrow R)$. Moreover, $R \odot R^* \leq R$ iff $(R \Leftarrow R)$. Moreover, $R^*$ is transitive iff $R \leq R \odot R$.

(6) $R$ is transitive iff $R \odot R \leq R$ iff $R \leq (R \Rightarrow R)$ iff $R \leq (R \Leftarrow R)$. Moreover, $R^*$ is transitive iff $R \leq R \odot R$.

(7) If $R^* \circ R^* \leq R^*$, then $R \leq R \odot R$.

(8) If $R$ is an $\odot$-quasi-equivalence relation, then $R = (R \circ R) = (R^* \Rightarrow R) = (R \Leftarrow R)^*$ and $R^* = R^* \odot R^*$. Moreover, $R \circ R$ and $R^* \circ R^*$ are symmetric.

(9) $R^* \circ R \leq R$ and $R$ is reflexive iff $R$ is an $\odot$-quasi-equivalence relation iff $(R \Rightarrow R)$ and $R$ are reflexive and $R \leq (R \Rightarrow R)$ iff $(R \Leftarrow R)$ and $R$ are reflexive and $R \leq (R \Leftarrow R)$.

(11) If $R^* \circ R^* \leq R$ and $R$ is reflexive, then $R = R \odot R^*$.

(12) If $R$ is an $\odot$-equivalence relation, then $R = (R \circ R) = (R \Rightarrow R) = (R \Leftarrow R)$ and $R^* = R^* \odot R^*$.

(13) If $R$ is reflexive and symmetric, then $R \Leftarrow R$ is an $\odot$-equivalence relation.

(14) Let $R$ be reflexive and symmetric. We define

$$R^\infty(x,y) = \bigvee_{n \in N} R^n(x,y)$$

where $R^n = R \circ R \circ \ldots \circ R$. Then $R^\infty$ is an $\odot$-equivalence relation.

(15) $(R \circ R^*)$ and $(R^* \circ R)$ are $\odot$-equivalence relations.

2. The various operations of fuzzy approximations

Definition 2.1. Let $(U, V, R)$ be an information space with a fuzzy relation $R \in L^U \times V$. For each $A \in L^U$ and $B \in L^V$, we define:

(1) A lower approximation $\text{apr}_{R_n} : L^V \to L^U$ is defined as:

$$\text{apr}_{R_n}(B)(x) = \bigvee_{y \in V} (R(x,y) \rightarrow B(y))$$

and an upper approximation $\overline{\text{apr}}_{R_n} : L^V \to L^U$ is defined as:

$$\overline{\text{apr}}_{R_n}(B)(x) = \bigwedge_{y \in V} (R(x,y) \odot B(y)).$$

(2) A lower approximation $\text{apr}_{R_n} : L^U \to L^V$ is defined as:

$$\text{apr}_{R_n}(A)(y) = \bigvee_{x \in U} (R(x,y) \rightarrow A(x))$$

and an upper approximation $\overline{\text{apr}}_{R_n} : L^U \to L^V$ is defined as:

$$\overline{\text{apr}}_{R_n}(A)(y) = \bigwedge_{x \in U} (R(x,y) \odot A(x)).$$

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(3) A map $Ib_R : L^V \rightarrow L^U$ is defined as:

$$Ib_R(B)(x) = \bigwedge_{y \in V} (B(y) \rightarrow R(x, y))$$

and a map $Ub_R : L^V \rightarrow L^U$ is defined as:

$$Ub_R(A)(y) = \bigwedge_{x \in U} (A(x) \rightarrow R(x, y)).$$

(4) A map $O_{R_u} : L^V \rightarrow L^U$ is defined as:

$$O_{R_u}(B)(x) = \bigwedge_{y \in V} (B(y) \oplus R(x, y))$$

and a map $O_{R_v} : L^U \rightarrow L^V$ is defined as:

$$O_{R_v}(A)(y) = \bigwedge_{x \in U} (A(x) \oplus R(x, y)).$$

(5) A map $P_{R_u} : L^V \rightarrow L^U$ is defined as

$$P_{R_u}(B)(x) = Ib^*_R(B)(x)$$

and a map $P_{R_v} : L^U \rightarrow L^V$ is defined as

$$P_{R_v}(A)(y) = Ub^*_R(A)(y).$$

**Theorem 2.2.** Let $(U, V, R)$ be an information space.

1. $\overline{apr}_{R_u}(B) = O_{R_u}(B) = Ib^*_R(B)$, for all $B \in L^V$.
2. $\overline{apr}_{R_v}(A) = O_{R_v}(A) = Ub^*_R(A)$, for all $A \in L^U$.
3. $(\overline{apr}_{R_u}(B))^* = Ib^*_R(B) = O_{R_v}(B)$
4. $(\overline{apr}_{R_v}(A))^* = Ub^*_R(A) = O_{R_u}(A)$
5. $O_{R_u}(B) = \overline{apr}_{R_u}(B) = Ib^*_R(B)$, for all $B \in L^V$.
6. $O_{R_v}(A) = \overline{apr}_{R_v}(A) = Ub^*_R(A)$, for all $A \in L^U$.
7. $P_{R_u}(B) = (\overline{apr}_{R_u}(B))^* = \overline{apr}_{R_v}(B)$, for all $B \in L^V$.
8. $P_{R_v}(A) = (\overline{apr}_{R_v}(A))^* = \overline{apr}_{R_u}(A)$, for all $A \in L^U$.

**Proof.** (1) From Lemma 1.3(13), we obtain:

$$\overline{apr}_{R_u}(B)(x) = \bigwedge_{y \in V} (R(x, y) \rightarrow B(y))$$

$$= \bigwedge_{y \in V} (R(x, y) \oplus B^*(y)) = O_{R_v}(B)(x)$$

$$\overline{apr}_{R_v}(A)(y) = \bigwedge_{x \in U} (A(x) \rightarrow R(x, y))$$

$$= \bigwedge_{y \in V} (A(x) \oplus R^*(y, x)) = P_{R_u}(A)(y).$$

(5) $O_{R_u}(B)(x) = \bigwedge_{y \in V} (B(y) \oplus R(x, y))$

Other cases are similarly proved.

**Theorem 2.3.** Let $(U, V, R_1)$ and $(V, W, R_2)$ be two information spaces. We have the following properties.

1. $\overline{apr}_{R_1 \circ R_2}(x) = \overline{apr}_{R_1} \circ \overline{apr}_{R_2}$
2. $\overline{apr}_{R_1 \circ R_2}(x) = \overline{apr}_{R_2} \circ \overline{apr}_{R_1}$
3. $Ib_{R_1 \circ R_2} = \overline{apr}_{R_1} \circ Ib_{R_2}$
4. $Ub_{R_1 \circ R_2} = Ub_{R_2} \circ \overline{apr}_{R_1}$
5. $Ib_{R_1 \circ R_2} = Ib_{R_1} \circ \overline{apr}_{R_2}$
6. $Ub_{R_1 \circ R_2} = \overline{apr}_{R_2} \circ Ub_{R_1}$
7. $O_{R_1 \circ R_2} = \overline{apr}_{R_2} \circ O_{R_1}$
8. $Ib_{R_1 \circ R_2} = \overline{apr}_{R_1} \circ Ib_{R_2}$
9. $Ub_{R_1 \circ R_2} = \overline{apr}_{R_1} \circ Ub_{R_2}$
10. $O_{R_1 \circ R_2} = O_{R_1} \circ O_{R_2}$
11. $P_{R_1 \circ R_2} = P_{R_1} \circ P_{R_2}$
12. $P_{R_1 \circ R_2} = \overline{apr}_{R_1} \circ (\overline{apr}_{R_2}(A))$
13. $P_{R_1 \circ R_2} = \overline{apr}_{R_1} \circ (\overline{apr}_{R_2}(A))$

**Proof.** (1) For each $C \in L^W$ and $x \in U$,

$$\overline{apr}_{R_1 \circ R_2}(C)(x) = \bigwedge_{y \in W} (R_1(y, x) \rightarrow C(z))$$

$$= \bigwedge_{y \in W} (R_1(y, x) \circ R_2(y, z) \rightarrow C(z))$$

$$= \bigwedge_{y \in W} (R_1(y, x) \rightarrow (R_2(y, z) \rightarrow C(z)))$$

$$= \bigwedge_{y \in W} (\overline{apr}_{R_1}(R_2(y, z) \rightarrow C(z)))$$

$$= \bigwedge_{y \in W} (\overline{apr}_{R_2}(C)(y)).$$
(3) For each \( C \in L^W \) and \( x \in U \),

\[
Ib_{(R_1 \Rightarrow_R_2)}(C)(x) = \bigwedge_{z \in W} (C(z) \rightarrow (R_1 \Rightarrow_R_2)(x, z)) = \bigwedge_{z \in W} \bigwedge_{y \in V} (R_1(x, y) \rightarrow R_2(y, z)) = \bigwedge_{z \in W} \bigwedge_{y \in V} (C(z) \circ R_1(x, y) \rightarrow R_2(y, z)) = \bigwedge_{z \in W} \bigwedge_{y \in V} (R_1(x, y) \rightarrow (C(z) \rightarrow R_2(y, z))) = \bigwedge_{z \in W} \bigwedge_{y \in V} R_1(x, y) \rightarrow \bigwedge_{z \in W} (C(z) \rightarrow R_2(y, z)) = \bigwedge_{z \in W} \bigwedge_{y \in V} R_1(x, y) \rightarrow \bigwedge_{z \in W} (C(z) \in U) = \bigwedge_{z \in W} R_1(x, y) \rightarrow Ib_{R_2}(C)(y) = \frac{pr}{R_1 \in [U]} (Ib_{R_2}(C))(x).
\]

(4) For each \( A \in L^U \) and \( z \in W \),

\[
Ub_{(R_1 \Rightarrow_R_2)}(A)(z) = \bigwedge_{x \in U} (A(x) \rightarrow (R_1 \Rightarrow_R_2)(x, z)) = \bigwedge_{x \in U} \bigwedge_{y \in V} (R_1(x, y) \rightarrow R_2(y, z)) = \bigwedge_{x \in U} \bigwedge_{y \in V} (A(x) \rightarrow (R_1(x, y) \rightarrow R_2(y, z))) = \bigwedge_{x \in U} \bigwedge_{y \in V} (A(x) \rightarrow (R_1(x, y) \circ R_2(y, z))) = \bigwedge_{x \in U} \bigwedge_{y \in V} (V \in U) (A(x) \circ R_1(x, y) \rightarrow R_2(y, z)) = \bigwedge_{x \in U} \bigwedge_{y \in V} \frac{pr}{R_1 \in [U]} (A)(y) \rightarrow R_2(y, z) = Ub_{R_2}(\frac{pr}{R_1 \in [U]} (A))(z).
\]

(5) For each \( C \in L^W \) and \( x \in U \),

\[
Ib_{(R_1 \Rightarrow_R_2)}(C)(x) = \bigwedge_{z \in W} (C(z) \rightarrow (R_1 \Rightarrow_R_2)(x, z)) = \bigwedge_{z \in W} \bigwedge_{y \in V} (R_1(x, y) \rightarrow R_2(y, z)) = \bigwedge_{z \in W} \bigwedge_{y \in V} (C(z) \rightarrow R_2(y, z)) = \bigwedge_{z \in W} \bigwedge_{y \in V} (C(z) \circ R_2(y, z)) = \bigwedge_{z \in W} \bigwedge_{y \in V} (R_1(x, y) \rightarrow (C(z) \circ R_2(y, z))) = \bigwedge_{z \in W} \bigwedge_{y \in V} R_1(x, y) \rightarrow \bigwedge_{z \in W} (C(z) \circ R_2(y, z)) = \bigwedge_{z \in W} \bigwedge_{y \in V} R_1(x, y) \rightarrow \bigwedge_{z \in W} (C(z) \circ R_2(y, z)) = \bigwedge_{z \in W} \bigwedge_{y \in V} (C(z) \circ R_2(y, z)) = \bigwedge_{z \in W} \bigwedge_{y \in V} \frac{pr}{R_1 \in [U]} (C)(y) \rightarrow R_1(x, y) = Ib_{R_1}(\frac{pr}{R_1 \in [U]} (C))(x).
\]

(6) For each \( A \in L^U \) and \( z \in W \),

\[
Ub_{(R_1 \Rightarrow_R_2)}(A)(z) = \bigwedge_{x \in U} (A(x) \rightarrow (R_1 \Rightarrow_R_2)(x, z)) = \bigwedge_{x \in U} \bigwedge_{y \in V} (R_1(x, y) \rightarrow R_2(y, z)) = \bigwedge_{x \in U} \bigwedge_{y \in V} (A(x) \rightarrow (R_1(x, y) \circ R_2(y, z))) = \bigwedge_{x \in U} \bigwedge_{y \in V} (R_2(y, z) \rightarrow (A(x) \circ R_1(x, y))) = \bigwedge_{x \in U} \bigwedge_{y \in V} (R_2(y, z) \rightarrow \bigwedge_{x \in U} (A(x) \circ R_1(x, y))) = \bigwedge_{x \in U} \bigwedge_{y \in V} \frac{pr}{R_1 \in [U]} (A)(y) \rightarrow R_1(x, y) = \frac{pr}{R_2 \in [U]} (Ub_{R_1}(A))(z).
\]

(8) For each \( C \in L^W \) and \( x \in U \),

\[
Ib_{R_1 \circ_R_2}(C)(x) = \bigwedge_{z \in W} (C(z) \rightarrow (R_1 \circ_R_2)(x, z)) = \bigwedge_{z \in W} \bigwedge_{y \in V} (R_1(x, y) \circ_R_2 R_2(y, z)) = \bigwedge_{z \in W} \bigwedge_{y \in V} (C(z) \circ R_1(x, y) \circ R_2(y, z)) = \bigwedge_{z \in W} \bigwedge_{y \in V} ((R_1(x, y) \circ R_2(y, z)) \rightarrow C(z)) = \bigwedge_{z \in W} \bigwedge_{y \in V} (R_1(x, y) \circ R_2(y, z) \rightarrow C(z)) = \bigwedge_{z \in W} \bigwedge_{y \in V} (R_1(x, y) \rightarrow \bigwedge_{z \in W} (C(z) \circ R_2(y, z))) = \bigwedge_{z \in W} \bigwedge_{y \in V} (R_1(x, y) \rightarrow \bigwedge_{z \in W} (C(z) \circ R_2(y, z))) = \bigwedge_{z \in W} \bigwedge_{y \in V} \frac{pr}{R_1 \in [U]} (R_2(C))(x) = \bigwedge_{z \in W} \bigwedge_{y \in V} \frac{pr}{R_1 \in [U]} (R_2(C))(x).
\]

(10) For each \( C \in L^W \) and \( x \in U \),

\[
O_{R_1}(O_{R_2}(C))(x) = \bigwedge_{z \in W} (O_{R_2}(C)(y) \rightarrow R_1(x, y)) = \bigwedge_{z \in W} \bigwedge_{y \in V} (O_{R_2}(C)(y) \rightarrow R_1(x, y)) = \bigwedge_{z \in W} \bigwedge_{y \in V} (O_{R_2}(C)(y) \rightarrow (R_1 \circ_R_2)(x, y)) = \bigwedge_{z \in W} \bigwedge_{y \in V} (O_{R_2}(C)(y) \rightarrow (R_1 \circ_R_2)(x, y)) = \bigwedge_{z \in W} \bigwedge_{y \in V} (O_{R_2}(C)(y) \rightarrow (R_1 \circ_R_2)(x, y)) = \bigwedge_{z \in W} \bigwedge_{y \in V} \frac{pr}{R_1 \in [U]} (O_{R_2}(C))(x).
\]

(11) For each \( A \in L^U \) and \( z \in W \),

\[
(P_{R_1 \circ_R_2}(P_{R_1}(A))) \ast(z) = \bigwedge_{y \in V} (P_{R_1}(A)(y) \rightarrow R_2(y, z)) = \bigwedge_{y \in V} \bigwedge_{y \in V} (P_{R_1}(A)(y) \rightarrow R_2(y, z)) = \bigwedge_{y \in V} \bigwedge_{y \in V} (P_{R_1}(A)(y) \rightarrow (R_1 \circ_R_2)(x, y)) = \bigwedge_{y \in V} \bigwedge_{y \in V} (P_{R_1}(A)(y) \rightarrow (R_1 \circ_R_2)(x, y)) = \bigwedge_{y \in V} \bigwedge_{y \in V} \frac{pr}{R_1 \in [U]} (P_{R_1}(A))(x).
\]

Other cases are similarly proved. \( \square \)

Example 2.4. Let \( U = \{x_1, x_2, x_3, x_4\}, V = \{y_1, y_2, y_3\}, W = \{z_1, z_2\} \) be sets and \( R_1 \in L^U \times V, R_2 \in L^W \times U \).
Define binary operations $\circ, \rightarrow$ (called Łukasiewicz conjunction) on $[0, 1]$ by
\[
x \circ y = \max\{0, x + y - 1\}, \quad x \rightarrow y = \min\{1 - x + y, 1\}.
\]
Then $([0, 1], \lor, \circ, 0, 1)$ is a stsc-quantale (ref.[10,11,17]).

\[
R_1 \circ R_2 = \begin{pmatrix}
1 & 0.6 & 0.8 \\
0.7 & 1 & 0.5 \\
0.3 & 0.6 & 1
\end{pmatrix}, \quad R_1 \rightarrow R_2 = \begin{pmatrix}
1 & 0 \\
0.7 & 1 \\
0.4 & 0.5
\end{pmatrix}.
\]

Put $C(z_1) = 0.8, C(z_2) = 0.7$. We obtain
\begin{enumerate}
    \item \(\text{ap}_{R_1 \circ R_2}(C) = (0.8, 0.7, 1, 0.9, 0.9) = \text{ap}_{R_1}(C) \circ \text{ap}_{R_2}(C)\).
    \item \(I_{b_{R_1 \circ R_2}}(C) = (0.3, 0.9, 0.6, 0.4) = \text{ap}_{R_1}(I_{b_{R_2}}(C))\).
    \item \(I_{b_{R_1 \circ R_2}}(C) = (0.3, 0.9, 0.6, 0.4) = \text{ap}_{R_1}(I_{b_{R_2}}(C))\).
    \item \(I_{b_{R_1 \circ R_2}}(C) = (0.3, 0.9, 0.6, 0.4) = \text{ap}_{R_1}(I_{b_{R_2}}(C))\).
\end{enumerate}

**Corollary 2.5.** Let $(U, U, R)$ be an information space. For $R \in L^{U \times U}$, we have the following properties.

\begin{enumerate}
    \item $F_{R_1} \circ F_{R_2} = F_{(R_1 \circ R_2)_{\lor}}$ and $F_{R_1} \circ F_{R_2} = F_{(R_1 \circ R_2)_{\land}}$, for each $F \in \{\text{ap}_{R_1}, \text{ap}_{R_2}\}$.
    \item $F_{R_1} \circ F_{R_2} = F_{(R_1 \circ R_2)_{\lor}}$ and $F_{R_1} \circ F_{R_2} = F_{(R_1 \circ R_2)_{\land}}$, for each $F \in \{\text{ap}_{R_1}, \text{ap}_{R_2}\}$.
    \item $F_{R_1} \circ F_{R_2} = F_{(R_1 \circ R_2)_{\lor}}$ and $F_{R_1} \circ F_{R_2} = F_{(R_1 \circ R_2)_{\land}}$, for each $F \in \{\text{ap}_{R_1}, \text{ap}_{R_2}\}$.
    \item $F_{R_1} \circ F_{R_2} = F_{(R_1 \circ R_2)_{\lor}}$ and $F_{R_1} \circ F_{R_2} = F_{(R_1 \circ R_2)_{\land}}$, for each $F \in \{\text{ap}_{R_1}, \text{ap}_{R_2}\}$.
\end{enumerate}

**Proof.** (2)

\[
\text{ap}_{R_1}(\text{ap}_{R_2}(A))(x) = \bigvee_{y \in Y} (\text{ap}_{R_1}(A)(y) \circ R(x, y)) = \bigwedge_{z \in U} (A(z) \circ \bigvee_{y \in Y} (R(z, y) \circ R(x, y)))
\]

Other cases are similarly proved from Theorem 2.3.

From Theorem 1.7 and Corollary 2.5, we can obtain the following corollaries.

**Corollary 2.6.** Let $R \in L^{U \times U}$ be an quasi-equivalence relation. Then we have the following properties.

\begin{enumerate}
    \item $F_{R_1} \circ F_{R_2} = F_{R_1}$ and $F_{R_1} \circ F_{R_2} = F_{R_2}$, for each $F \in \{\text{ap}_{R}, \text{ap}_{R}\}$.
    \item $I_{b_{R_1}} = \text{ap}_{R_1} \circ I_{b_{R_2}}$.
    \item $U_{b_{R_1}} = U_{b_{R_2}} \circ \text{ap}_{R_2}$.
    \item $I_{b_{R_2}} = I_{b_{R_1}} \circ \text{ap}_{R_2}$.
    \item $U_{b_{R_2}} = U_{b_{R_1}} \circ \text{ap}_{R_2}$.
    \item $O_{R_1} = \text{ap}_{R_1} \circ O_{R_2}$ and $O_{R_1} = O_{R_2}$.
\end{enumerate}

Moreover, if $R$ is symmetric, then $F_{R_1} \circ F_{R_2} = F_{R_1} \circ F_{R_2}$, for each $F \in \{\text{ap}_{R}, \text{ap}_{R}\}$.

**Corollary 2.7.** Let $R^* \in L^{U \times U}$ be an quasi-equivalence relation. Then we have the following properties.

\begin{enumerate}
    \item $F_{R_1} \circ F_{R_2} = F_{R_1}$ and $F_{R_1} \circ F_{R_2} = F_{R_2}$, for each $F \in \{O, P\}$.
    \item $I_{b_{R_1}} = \text{ap}_{R_1} \circ I_{b_{R_2}}$.
    \item $U_{b_{R_1}} = U_{b_{R_2}} \circ \text{ap}_{R_2}$.
    \item $I_{b_{R_2}} = I_{b_{R_1}} \circ \text{ap}_{R_2}$.
    \item $U_{b_{R_2}} = U_{b_{R_1}} \circ \text{ap}_{R_2}$.
    \item $O_{R_1} = \text{ap}_{R_1} \circ O_{R_2}$ and $O_{R_1} = O_{R_2}$.
\end{enumerate}

Moreover, if $R$ is symmetric, then $F_{R_1} \circ F_{R_2} = F_{R_1} \circ F_{R_2}$, for each $F \in \{O, P\}$.

**Example 2.8.** Let $(U = \{a, b, c\}, AT = V = \{a, b, c\}, R)$ be an information space as follows:

\[
R = \begin{pmatrix}
1 & 0.4 & 0.1 \\
0 & 1.0 & 0.3 \\
0 & 0.5 & 1.0
\end{pmatrix}.
\]

Define binary operations $\circ, \rightarrow$ as same as in Example 2.4.

(1) Since $R \circ R = R$ and $R$ is reflexive, we have properties (1-6) of Corollary 2.6.

(2) We have $R \circ R^* \neq R$ because

\[
0 = R(b, a) \neq \bigvee_{y \in Y} (R(b, y) \circ R(a, y)) = 0.4
\]

\[
R \circ R^* = \begin{pmatrix}
1 & 0.4 & 0.1 \\
0.4 & 1.0 & 0.5 \\
0.1 & 0.5 & 1.0
\end{pmatrix}.
\]
For $A \in L^U$ with $A(a) = 0.1$, $A(b) = 0.9$, $A(c) = 0.2$, we have

\[
\mathcal{ap}_{R_c}(A)(a) = 0.1, \mathcal{ap}_{R_c}(A)(b) = 0.9, \mathcal{ap}_{R_c}(A)(c) = 0.2 \\
\mathcal{ap}_{R_c}(\mathcal{ap}_{R_c}(A))(a) = 0.3, \mathcal{ap}_{R_c}(\mathcal{ap}_{R_c}(A))(b) = 0.9, \mathcal{ap}_{R_c}(\mathcal{ap}_{R_c}(A))(c) = 0.4.
\]

Hence $\mathcal{ap}_{R \cap R_c}(A) = \mathcal{ap}_{R_c}(\mathcal{ap}_{R_c}(A)) \neq \mathcal{ap}_{R_c}(A)$.

(3) For $B \in L^U$ with $B(a) = 0.1, B(b) = 0.5, B(c) = 0.9$, we have

\[
\mathcal{ap}_{R_a}(B)(a) = \bigwedge_{y \in V} \mathcal{R}(a, y) \rightarrow B(y) = 0.1, \\
\mathcal{ap}_{R_a}(B)(b) = 0.5, \mathcal{ap}_{R_a}(B)(c) = 0.9
\]

Since

\[
\mathcal{ap}_{R_c}(\mathcal{ap}_{R_a}(B))(c) = \bigwedge_{x \in V} (R(x, a) \rightarrow \mathcal{ap}_{R_a}(B)(x)) = 0.8
\]

we have $\mathcal{ap}_{R \circ R_c}(B) = \mathcal{ap}_{R_c}(\mathcal{ap}_{R_a}(B)) \neq \mathcal{ap}_{R_a}(B)$.

References


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