THE BASES OF PRIMITIVE NON-POWERFUL COMPLETE SIGNED GRAPHS

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Abstract. The base of a signed digraph \( S \) is the minimum number \( k \) such that for any vertices \( u, v \) of \( S \), there is a pair of walks of length \( k \) from \( u \) to \( v \) with different signs. Let \( K \) be a signed complete graph of order \( n \), which is a signed digraph obtained by assigning \(+1\) or \(-1\) to each arc of the \( n \)-th order complete graph \( K_n \) considered as a digraph. In this paper we show that for \( n \geq 3 \) the base of a primitive non-powerful signed complete graph \( K \) of order \( n \) is 2, 3 or 4.

1. Introduction

A sign pattern matrix \( M \) is a square matrix with entries in \( \{1, 0, -1\} \). In multiplying two sign pattern matrices, we use the operating rules of entries that continues to hold the signs of the usual addition and multiplication, that is

\[
1+1 = 1; \quad (-1)+(-1) = -1; \quad 1+0 = 0+1 = 1; \quad (-1)+0 = 0+(-1) = -1; \\
0\cdot a = a\cdot 0 = 0; \quad 1\cdot (-1) = -1; \quad (-1)\cdot (-1) = 1; \quad 1\cdot (-1) = (-1)\cdot 1 = -1 \quad \text{for any} \quad a \in \{1, 0, -1\}.
\]
In this case we contact the ambiguous situations $1 + (-1)$ and $(-1) + 1$, which we will use the notation "♯" as in [2]. Define the addition and multiplication which involving the symbol "♯" as follows: For any $a \in \Gamma = \{1, 0, -1, \#\}$,

\[
(-1) + 1 = 1 + (-1) = \#; \quad a + \# = \# + a = \#
\]

\[
0 \cdot \# = \# \cdot 0 = 0; \quad a \cdot \# = \# \cdot a = \# \quad \text{(when } a \neq 0)\]

Matrices with entries in \(\Gamma\) are called \textit{generalized sign pattern matrices}. The addition and multiplication of the entries of generalized sign pattern matrices are defined in the usual way such that they coincide with the operations in sign pattern matrices.

**Definition 1.** A square generalized sign pattern matrix \(M\) is \textit{powerful} if each power of \(M\) contains no \# entry. A square generalized sign pattern matrix \(M\) is called \textit{non-powerful} if it is not powerful.

**Definition 2.** Let \(M\) be a square generalized sign pattern matrix of order \(n\). The smallest number \(l\) such that \(M^l = M^{l+p}\) for some \(p\) is called the \textit{(generalized) base} of \(M\) and denoted by \(l(M)\). The least positive integer \(p\) such that \(M^l = M^{l+p}\) for \(l = l(M)\) is called to be the \textit{(generalized) period} of \(M\) and denote it by \(p(M)\).

We introduce some graph theoretic concepts of generalized sign pattern matrices.

A \textit{signed digraph} \(S = (V, A, f)\) is a digraph with vertex set \(V\), arc set \(A\) and a sign function \(f\) defined on \(A\) with its value \(1, -1\). For \(v, w \in V\) we say \(f(vw)\) the \textit{sign} of an arc \(vw\), and we denote it by \(\text{sgn}(vw)\). The \textit{sign} of a (directed) walk \(W\) in \(S\), denoted by \(\text{sgn}(W)\) or \(f(W)\), is the product of signs of all arcs in \(W\). For example if \(W = v_1v_2v_3v_4\), then \(\text{sign}(W) = f(W) = f(v_1v_2)f(v_2v_3)f(v_3v_4)\). If two walks \(W_1\) and \(W_2\) have the same initial points, the same terminal points, the same lengths and different signs, then we say that \(W_1\) and \(W_2\) are a \textit{pair of SSSD walks}.

A (signed) digraph \(S\) is \textit{primitive} if there is a positive integer \(k\) such that for all vertices \(v, w\) of \(S\) there is a walk of length \(k\) from \(v\) to \(w\). A signed digraph \(S\) is \textit{powerful} if \(S\) contains no pair of SSSD walks. Also \(S\) is \textit{non-powerful} if it is not powerful. Hence every non-powerful primitive signed digraph contains a pair of SSSD walks. Let \(M = M(S) = [a_{ij}]\) be the adjacency matrix of a signed digraph \(S\), that is, the arc \((i, j)\) has sign \(\text{sgn}(i, j) = \alpha\) if and only if \(a_{ij} = \alpha\) with \(\alpha = 1\), or \(-1\). Hence
the adjacency (signed) matrix $M$ of a signed digraph $S$ is a sign pattern matrix which satisfies that the $(i, j)$-entry of $M^k$ is 0 if and only if $S$ contains no walk of length $k$ from $i$ to $j$. Also $(i, j)$-entry of $M^k$ is 1 (or $-1$) if and only if all walks of length $k$ from $i$ to $j$ in $S$ are of sign 1 (or, $-1$). The $(i, j)$-entry of $M^k$ is $\#$ if and only if $S$ contains a pair of SSSD walks of length $k$ from $i$ to $j$. We see from the above relations between matrices and digraphs that each power of a signed digraph $S$ contains no pair of SSSD walks if and only if the adjacency matrix $M$ is powerful. Henceforth we may also say that a signed digraph $S$ is powerful or non-powerful if its adjacency sign pattern matrix $M$ is powerful or non-powerful respectively.

From now on we assume that $S = (V, A, f)$ is a primitive non-powerful signed digraph of order $n$. For each pair of vertices $u, v$ of $S$, we define the local base $l_S(u, v)$ from $u$ to $v$ to be the smallest integer $l$ such that for each $k \geq l$, there is a pair of SSSD walks of length $k$ from $u$ to $v$ in $S$. The base $l(S)$ of $S$ is defined to be $\max\{l_S(u, v) | u, v \in V(S)\}$. It follows directly from the definitions that $l(S) = l(M)$ where $M$ is the adjacency matrix of $S$.

The upper bounds for the bases of primitive non-powerful sign pattern matrices are found by You et al. [5]. They also characterized extremal cases completely. Gao et al.[1], Shao and Gao[4] and Li and Liu [3] studied the base and the local base of a primitive non-powerful signed symmetric digraphs with loops.

Let us assume that $K$ is a complete non-powerful signed digraph of order $n$ which is the $n$-th order complete graph (considered as a digraph) by assigning signs to each arc such that it becomes a non-powerful signed digraph. In this paper we prove that the base of $K$ is less than or equals to 4. As a consequence if all the entries of a non-powerful sign pattern matrix $A$ are nonzero except diagonals, then the all entries of $A^4$ are $\#$. We also provide the examples when the base of $K$ is 2, 3 and 4 respectively.

2. Main theorems

Let $K = (V, A, f)$ be a complete non-powerful signed digraph of order $n$. That is, $K$ is the $n$-th order digraph which has unique arc for each ordered pair of vertices of $K$ and signs are assigned to each arc such that $K$ becomes a non-powerful signed digraph. Let $v_1, v_2, \cdots, v_r$,
be vertices of $K$. If $C$ is a directed walk from $v_1$ to $v_r$ which goes through $v_2, v_3, \cdots v_{r-1}$, then we denote $C$ by $v_1v_2\cdots v_{r-1}v_r$ and the sign $f(v_1v_2)f(v_2, v_3)\cdots f(v_{r-1}v_r)$ of $C$ by $f(C) = f(v_1v_2\cdots v_{r-1}v_r) = \text{sgn}(C) = \text{sgn}(v_1v_2\cdots v_{r-1}v_r)$. Throughout this paper we use the notation $u \xrightarrow{k} v$ if there is a walk of length $k$ from a vertex $u$ to another vertex $v$. The sum $W_1 + W_2$ of two walks $W_1 = v_1v_2\cdots v_n$ and $W_2 = w_1w_2\cdots w_m$ such that $v_n = w_1$ and the inverse $-W_1$ of $W_1$ are defined by $W_1 + W_2 = v_1v_2\cdots v_nw_2w_3\cdots w_m$ and $-W_1 = v_nv_{n-1}\cdots v_1$.

**Theorem 1.** The base $l(K)$ of the complete non-powerful signed digraph $K$ of order $n \geq 4$ is less than or equals to 4.

**Proof.** It suffices to show that there is a pair of SSSD walks of common length 4 from $u$ to $v$. Let $u, v$ be vertices of $K$. Since $n \geq 4$, we can choose a vertex $w$ of $K$ different from $u$ and $v$. Let $\sigma$ be the sign of the walk $uvw$. If there is a vertex $x$ of $K$ such that $x \neq u$ and the sign of the walk $uxu$ is $-\sigma$, then $uvwuv$ and $uxuwv$ are a pair of SSSD walks of length 4 from $u$ to $v$.

If the sign of the walk $uxu$ is $\sigma$ for any vertex $x$ of $K$ and there are distinct vertices $y, z$ of $K$ such that $z \neq u$ and the sign of the walk $yzy$ is $-\sigma$, then both $y$ and $z$ are different from $u$. If $y \neq v$, then $uuuv$ and $uyzuv$ are a pair of SSSD walks with common length 4 from $u$ to $v$.

If $y = v$, then since $z \neq v$, $uwuzv$ and $uzyzv$ are desired pair of SSSD walks with common length 4 from $u$ to $v$.

Assume that the sign of the walk $yzy$ is $\sigma$ for all distinct vertices $y, z$.

If $\sigma = -1$, then

$$\text{sgn}(uvw)\text{sgn}(uwuv) = f(uv)f(vw)f(wu)f(ww)f(uw)f(wu)f(vu)f(uv)$$

$$= (f(ww)f(vw))(f(vw)f(uv))(f(ww)f(uw))(f(uv))^2$$

$$= \text{sgn}(uvu)\text{sgn}(vvw)\text{sgn}(uwu) = \sigma^3 = -1.$$ 

Hence $uvwuv$ and $uwvuv$ are a pair of SSSD walks with common length 4 from $u$ to $v$.

If $\sigma = 1$, then since $K$ is non-powerful, there is an even cycle of sign $-1$, or there are two odd cycles with different signs. Assume that there is an even cycle $x_1x_2\cdots x_kx_1$ with sign $-1$. If $x_i \neq u$ for all $i = 1, 2, \cdots, k$, then
then since
\[
\text{sgn}(ux_1x_2u)\text{sgn}(ux_2x_3u)\cdots\text{sgn}(ux_{k-1}x_{k}u)\text{sgn}(ux_kx_1u) = (f(ux_1)f(x_1x_2)f(x_2u))(f(ux_2)f(x_2x_3)f(x_3u))
\]
\[
\cdots(f(ux_{k-1})f(x_{k-1}x_k)f(x_ku))(f(ux_k)f(x_kx_1)f(x_1u)) = f(x_1x_2)f(x_2x_3)\cdots f(x_{k-1}x_k)f(x_kx_1)
\]
\[
= \text{sgn}(x_1x_2\cdots x_kx_1) = -1,
\]

among the walks \(ux_1x_2u, ux_2x_3u, \cdots, ux_{k-1}x_ku, ux_kx_1u,\) there are two walks \(C_1, C_2\) with different signs. Thus \(C_1 + uv\) and \(C_2 + uv\) are a pair of SSSD walks SSSD walks of common length 4 from \(u\) to \(v\).

Let us assume that there are two odd cycles \(y_1y_2\cdots y_i y_1\) and \(z_1z_2\cdots z_m z_1\) with signs 1 and \(-1\) respectively. We want to show that there is a walk \(C_3 = uy_iy_{i+1}u\) (or \(C_3 = uy_iy_1u\)) of sign \(+1\). If \(u \neq y_i\) for all \(i = 1, 2, \cdots, l\), then since
\[
\text{sgn}(uy_1y_2u)\text{sgn}(uy_2y_3u)\cdots\text{sgn}(uy_{i-1}y_iu)\text{sgn}(uy_iy_1u) = \text{sgn}(y_1y_2\cdots y_{i}y_1) = 1,
\]

among the walks \(uy_1y_2u, uy_2y_3u, \cdots, uy_{i-1}y_iu, uyy_1u,\) there is a walk \(C_3\) with sign \(+1\). If \(u = y_i\) for some \(i\), then since
\[
\text{sgn}(uy_1y_2u)\text{sgn}(uy_2y_3u)\cdots\text{sgn}(uy_{i-2}y_{i-1}u)
\]
\[
\text{sgn}(uy_{i+1}y_{i+2}u)\cdots\text{sgn}(uy_{i-1}y_{i}u)\text{sgn}(uy_iy_1u) = \text{sgn}(y_1y_2\cdots y_{i}y_1) = 1,
\]

we have a walk from \(u\) to \(v\) of length 3 with sign 1.

Similarly among the walks \(uz_1z_2u, uz_2z_3u, \cdots, uz_{m-1}z_mu, uz_mz_1u,\) there is a walk \(C_4\) of sign \(-1\). Thus \(C_3 + uv\) and \(C_4 + uv\) are a pair of SSSD walks with common length 4 from \(u\) to \(v\). As a consequence, we have \(l(K) \leq 4\).

We will show the upper bound 4 in Theorem 1 is extremal by constructing a complete nonpowerful signed digraph of base at least 4.
Theorem 2. Let $V = \{v_1, v_2, \ldots, v_n\}$, $A = \{(v_i, v_j)|1 \leq i, j \leq n, i \neq j\}$ and $f : A \rightarrow \{-1, 1\}$ such that
\[
f(v_i, v_j) = \begin{cases} 
-1, & \text{if } j = 3 \text{ and } i \neq 1, \text{ or } (i, j) = (3, 2); \\
1, & \text{otherwise.}
\end{cases}
\]
The signed digraph $G = (V, A, f)$ is primitive non-powerful and $l(G) \geq 4$.

Proof. Let $W$ be a walk of length 3 from $v_1$ to $v_2$. Then $W = v_1v_iv_jv_2$ for some $i, j$. If $i = 2$, then for all $j \neq 2$ since $f(v_2v_jv_2) = f(v_2v_j)f(v_jv_2) = 1$, we have $\text{sgn}(v_1v_2v_jv_2) = 1$. If $i = 3$, then $j \neq 3$. Hence $f(v_1v_3v_jv_2) = f(v_1v_3)f(v_3v_j)f(v_jv_2) = 1$. If $i \geq 4$ and $j = 3$, then $\text{sgn}(v_1v_3v_4v_2) = f(v_1v_3)f(v_3v_4)f(v_4v_2) = 1(-1)(-1) = 1$. If $i \geq 4$ and $j \neq 3$, then $\text{sgn}(v_1v_iv_jv_2) = f(v_1v_i)f(v_i)v_jf(v_jv_2) = 1$. Hence the sign of a walk of length 3 from $v_1$ and $v_2$ is always 1. We have $l(v_1, v_2) \geq 4$, and hence $l(G) \geq 4$. By Theorem 1, we conclude that $l(G) = 4$. \qed

We can easily see that the base of a primitive non-powerful digraph is at least 2. In the following examples we provide two complete signed graphs of order $n \geq 4$ with base 2 and 3 respectively. As a result, the possible base of a complete signed graph of order $n \geq 4$ is 2, 3 and 4.

Example 1. Let $n \geq 4$, $V = \{v_1, v_2, \ldots, v_n\}$, $A = \{(v_i, v_j)|1 \leq i, j \leq n, i \neq j\}$ and $f : A \rightarrow \{-1, 1\}$ such that
\[
f(v_iv_j) = \begin{cases} 
-1, & \text{if } j = 3 \text{ and } i \neq 1, \text{ or } (i, j) = (3, 2), \text{ or } (i, j) = (1, 2), \\
1, & \text{otherwise}
\end{cases}
\]
We find a pair of SSSD walks of length 2 from $v_i$ to $v_j$ as follows for each $i$ and $j$.
\[
\begin{align*}
&v_1v_2v_1 \quad \text{if } i = 1 \text{ and } j = 2, \\
v_1v_3v_2 \quad \text{if } i = 1 \text{ and } j = 2, \\
v_1v_2v_3 \quad \text{if } i = 1 \text{ and } j = 3, \\
v_1v_2v_j \quad \text{if } i = 1 \text{ and } j \geq 4, \\
v_2v_3v_1 \quad \text{if } i = 2 \text{ and } j = 1, \\
v_2v_1v_2 \quad \text{if } i = 1 \text{ and } j = 2, \\
v_2v_1v_3 \quad \text{if } i = 2 \text{ and } j = 3, \\
v_2v_1v_j \quad \text{if } i = 2 \text{ and } j \geq 4,
\end{align*}
\]
As a consequence, the signed digraph $G = (V, A, f)$ is primitive non-powerful and $l(G) = 2$.

**Example 2.** Let $n \geq 4$, $V = \{v_1, v_2, \ldots, v_n\}$, $A = \{(v_i, v_j) | 1 \leq i, j \leq n, i \neq j\}$ and $f : A \rightarrow \{-1, 1\}$ such that

$$f(v_i v_j) = \begin{cases} -1, & \text{if } (i, j) = (1, 2), \\ 1, & \text{otherwise} \end{cases}.$$ 

We can see for each walk of length 2 from $v_1$ to $v_2$ is of sign +1. Thus $l(G) \geq 3$. By the same method used in above example, there are a pair of SSSD walks of length 3 from $v_i$ to $v_j$ as follows for each $i$ and $j$. It follows that the signed digraph $G = (V, A, f)$ is primitive non-powerful and $l(G) = 3$.

A consequence of the above theorems and examples is that the base of a sign pattern matrix such that every diagonal entry is zero and every non diagonal entries is of sign 1 or -1 is 2, 3 and 4. Also we can consider the sign pattern matrix without zero entries. The corresponding digraph is a complete graph with loops on each vertices. In this case we have the following theorem.

**Theorem 3.** If $n \geq 3$ and $K$ is a non-powerful signed digraph over $n$-th order complete graph with loops on each vertices, then $l(K) \leq 3$.

**Proof.** Suppose that $l(K) \geq 4$. There are $v, w \in V$ and $\sigma \in \{+1, -1\}$ such that the sign of every walk from $v$ to $w$ of length 3 is always $\sigma$. Let $\tau$ be the sign of the loop incident on $v$. For all $x \in V$, since $\sgn(vvwx) = \sgn(vv)\sgn(vwx) = \tau \sgn(vwx) = \sigma$, we have $\sgn(xxw) = \sigma$. Since $\sgn(vxxw) = f(vx)f(xx)f(xw) = f(xx)\sgn(vwx) = f(xx)\sigma \tau = \sigma$, we have $f(xx) = \tau$. 

\begin{align*}
v_3v_2v_1 & \quad \text{if } i = 3 \text{ and } j = 1, \\
v_3v_1v_2 & \quad \text{if } i = 1 \text{ and } j = 2, \\
v_3v_1v_3 & \quad \text{if } i = 3 \text{ and } j = 2, \\
v_3v_1v_j & \quad \text{if } i = 3 \text{ and } j \geq 4, \\
v_1v_2v_1 & \quad \text{if } i = 1 \text{ and } j = 1, \\
v_1v_1v_2 & \quad \text{if } i \geq 4 \text{ and } j = 2, \\
v_1v_3v_2 & \quad \text{if } i \geq 4 \text{ and } j = 3, \\
v_1v_1v_j & \quad \text{if } i \geq 4 \text{ and } j \geq 4.
\end{align*}
Let $C = x_1x_2 \cdots x_kx_1$ be a cycle of length $k$ in $K$. We have
\[
\sigma^k = \text{sgn}(vx_1x_2w)\text{sgn}(vx_2x_3w) \cdots \text{sgn}(vx_kx_1w)
= (f(vx_1)f(x_1x_2)f(x_2x_3)f(x_3x_4)) \cdots (f(vx_k)f(x_kx_1)f(x_1x_2))
= (f(vx_1)f(x_1x_2)) (f(vx_2)f(x_2x_3)) \cdots (f(vx_k)f(x_kx_1))
= (\sigma \tau)^k \text{sgn}(x_1x_2 \cdots x_kx_1)
= \sigma^k \tau^k f(C).
\]
Thus the signs of all even and odd cycles are 1 and $\tau$ respectively. Therefore $K$ is powerful. This is a contradiction. Hence $l(K) \leq 3$.

\[\square\]

Remark 1. Let $n = 3$, $V = \{v_1, v_2, v_3\}$ and $A = \{(v_i, v_j)| i \neq j\}$. Since $v_1v_3v_2$ is the only $v_1 \stackrel{2}{\rightarrow} v_2$ walk in $K$, we have $l(K) \geq 3$.
If $\text{sgn}(v_1v_2v_3) = \text{sgn}(v_2v_3v_2) = \text{sgn}(v_3v_1v_3) = 1$, then every 2-cycle in $G$ is of sign 1.
Since
\[
\text{sgn}(v_1v_2v_3v_1)\text{sgn}(v_1v_3v_2v_1)
= f(v_1v_2)f(v_2v_3)f(v_3v_1)f(v_1v_3)f(v_3v_2)f(v_2v_3)f(v_3v_1)
= (f(v_1v_2)f(v_2v_3)f(v_3v_1)f(v_1v_3)f(v_3v_2)f(v_2v_3)f(v_3v_1))^{'}
= \text{sgn}(v_1v_2v_1)\text{sgn}(v_2v_3v_2)\text{sgn}(v_3v_1v_3) = 1
\]
all 3-cycles in $K$ are of the same sign. It follows that $K$ is powerful.
If $\text{sgn}(v_1v_2v_3) = \text{sgn}(v_2v_3v_2) = \text{sgn}(v_3v_1v_3) = -1$ for all $v_i, v_j \in V$, then there is a $v_i \stackrel{2}{\rightarrow} v_j$ walk $W$ in $K$. Since
\[
\text{sgn}(v_1v_2v_3v_1)\text{sgn}(v_1v_3v_2v_1) = f(v_1v_2v_1)f(v_2v_3v_2)f(v_3v_1v_3) = -1,
\]
there are two $v_i \stackrel{3}{\rightarrow} v_i$ walks $W_1$ and $W_2$ in $K$ with different signs. Thus we see that $W + W_1$ and $W + W_2$ are a pair of SSSD walks with length 5. We have $l(K) \leq 5$. Let $W = w_0w_1w_2w_3w_4$ be a $v_1 \stackrel{4}{\rightarrow} v_1$ walk in $K$. Hence we have $w_0 = w_4 = v_1$. We may assume that $w_1 = v_2$. If $w_2 = v_1$, then $f(W) = f(v_1v_2v_1)f(v_1w_4v_1) = 1$. If $w_2 = v_3$, then since $w_3 = v_2$, we have $f(W) = 1$. Therefore there is no $v_1 \stackrel{4}{\rightarrow} v_1$ walk in $K$ with sign $-1$. Thus $l(K) = 5$.
If the signs of $f(v_1v_2v_1), f(v_2v_3v_2)$ and $f(v_3v_1v_3)$ are not equal, then we may assume that $f(v_1v_2v_1) = f(v_2v_3v_2) = -f(v_3v_1v_3)$. Let $v_i, v_j \in V$. Hence there is a $v_i \stackrel{2}{\rightarrow} v_j$ walk $W = v_i v_k v_j$ in $K$. If $i \neq 2$, then there
are two $v_i \xrightarrow{2} v_i$ walks $W_1$ and $W_2$ in $K$ with different signs. It is clear that $W_1 + W$ and $W_2 + W$ are a pair of SSSD walks with length 4. Similarly, we have a pair of SSSD walks with length 4 for the case $j \neq 2$. If $i = j = 2$, then $k \neq 2$. Whence there are a pair of $v_k \xrightarrow{2} v_k$ walks $X_1$ and $X_2$ in $K$ with different signs. Thus we see that $(v_i v_k) + X_1 + (v_k v_j)$ and $(v_i v_k) + X_2 + (v_k v_j)$ are a pair of SSSD walks with length 4. Hence $l(K) \leq 4$.

Let $f_1, f_2 : A \rightarrow \{1, -1\}$,

\[
f_1(v_i v_j) = \begin{cases} 
-1, & i = 1 \text{ and } j = 2 \\
1, & \text{otherwise},
\end{cases}
\]

and

\[
f_2(v_i v_j) = \begin{cases} 
-1, & i = 1 \text{ and } j = 2, 3 \\
1, & \text{otherwise}.
\end{cases}
\]

Then $(V, A, f_1)$ and $(V, A, f_2)$ are examples of signed digraph over complete graphs with loops with bases 3 and 4 respectively. Hence the possible bases of signed digraph over complete graphs with loops on 3 vertices are 3, 4 and 5.

Note that if

\[
A = \begin{pmatrix} 
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{pmatrix},
\]

then

\[
A^4 = \begin{pmatrix} 
1 & # & # \\
# & 1 & # \\
# & # & 1
\end{pmatrix}.
\]

References


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