EXPLICIT EVALUATIONS OF SPECIAL MULTIPLE ZETA VALUES, $\zeta(\{4l + 2\}_n)$ AND $\zeta(\{4l\}_n)$

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ABSTRACT. In this paper we calculate two special types of multiple zeta values, $\zeta(\{4l + 2\}_n)$ and $\zeta(\{4l\}_n)$ using the primitive roots of unity, which may be simpler and easier.

1. Introduction

The Euler-Zagier’s multiple zeta values is defined by

$$\zeta(s_1, s_2 \cdots, s_k) = \sum_{0 < m_1 < m_2 < \cdots < m_k} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_k^{s_k}}$$

with complex variables $s_i \ (i = 1, 2, \cdots, k)$. When $\Re(s_i) \geq 1$ for $i = 1, 2, \cdots, k - 1$ and $\Re(s_k) > 1$, multiple zeta values are absolutely convergent.

These values occur in the knot theory and quantum field theory. M. E. Hoffman [2] studied some relations and presented two conjectures, so called, sum and duality conjectures. These are first proved by Zagier [5] and extensively studied and generalized by Ohno [4].

One of the remarkable properties of the Riemann zeta function $\zeta(s)$ is that $\zeta(2n)$ can be evaluated in terms of the Bernoulli numbers as follows:

For a nonnegative integer $n$,

$$\zeta(2n) = (-1)^{n+1} \frac{2^{2n-1} B_{2n}}{(2n)!} \times \pi^{2n},$$

which is due to L. Euler.

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There are deep relations between multiple zeta values and Bernoulli numbers,
\[
\zeta(2, 2, \cdots, 2) = \frac{1}{m_1^2 m_2^2 \cdots m_k^2} = \frac{\pi^{2k}}{(2k + 1)!} \\
= (2\pi)^{2k} \sum_{m_1+2m_2+\cdots+km_k=k} \frac{1}{m_1! \cdots m_k!} \left( \frac{B_{2}}{2 \cdot 2!} \right)^{m_1} \cdots \left( \frac{B_{2k}}{2k(2k)!} \right)^{m_k}
\]
which are due to M. E. Hoffman [2]. J. M. Borwein and D. M. Bradley and D. J. Broadhurst [1] recommended further study of multiple zeta functions and other related functions. Here, we consider the case \((2l, \cdots, 2l)\) for indices \((s_1, \cdots, s_n)\) of the multiple zeta values. Let \(\mathbb{N}\) be the set of positive integers and \(\mathbb{Z}_+ = \mathbb{N} \cup \{0\}\), and let \(S(a; k)\) be the set of compositions of \(k\) length of \(k\), i.e.,
\[
S(a; k) = \{(a_1, \cdots, a_k)| a_1 + 2a_2 + \cdots + ka_k = k, \quad a_j \in \mathbb{Z}_+ \text{ for all } j = 1, 2, \cdots, k\}.
\]
The aim of this paper is to prove the case \((2l, \cdots, 2l)\) for indices \((s_1, \cdots, s_n)\) of the Euler-Zagier's multiple zeta values which generalizes the following two theorems.

We denote \(n\) repetitions of a substring by \(\{\cdots\}_n\) and \(\omega_l\) the \(l\)-th primitive root of unity.

Now, we will state our results.

**Theorem 1.** For \(n \in \mathbb{N}\) and \(l \in \mathbb{Z}_+\) we have
\[
\zeta(\{4l+2\}_n) = (2\pi)^{4ln+2n} \sum_{1 \leq k \leq 2l} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l}} \times \left( 1 + \sum_{k=1}^{2l} (-1)^{i_k} \omega_{2l+1}^{i_k} \right) \times \prod_{j=1}^{2ln+l+n} \frac{1}{a_j!} \left( \frac{B_{2j}}{2j(2j)!} \right)^{a_j}
\]
where the Bernoulli numbers \(B_j\) is defined by \(\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!}\).

This theorem can be regarded as a generalization of Hoffman [2], if \(l = 0\) then we obtain the Proposition 2.4 in Hoffman [2].
Theorem 2. For \( n, l \in \mathbb{N} \) we have

\[
\zeta\left(\{4l\}_n\right) = (-1)^{n+l} 2i(2\pi)^{4ln} \times \sum_{1 \leq k \leq 2l-1} \sum_{i_k=0}^{1} \left(1 + \sum_{k=1}^{2l-1} (-1)^{i_k} \omega_{4l}^k\right)^{2l(2n+1)} \times \prod_{(a_1, \ldots, a_{2ln+l}) \in \mathcal{S}(a;2ln+l)} \frac{1}{a_j!} \left(\frac{G_{2j}}{4j(2j)!}\right)^{a_j},
\]

where the Genocchi number \( G_j \) is defined by \( \frac{2te^{t}}{e^{2t}+1} = \sum_{j=0}^{\infty} G_j \frac{t^j}{j!} \) [3].

2. Proof of results

Lemma 1. Let \( n \in \mathbb{Z}_+ \), \( \alpha_i \in \mathbb{C} \) for all \( i \), and let \( [x] \) be the greatest integer not exceeding \( x \).

(1) If \( n \) is odd integers, then

\[
\prod_{i=1}^{n} \sin \alpha_i = \frac{(-1)^{[\frac{n}{2}]}}{2^{n-1}} \sum_{2 \leq k \leq n} \sum_{i_k=0}^{1} (-1)^{i_2+\cdots+i_n} \sin \left(\alpha_1 + \sum_{k=2}^{n} (-1)^{i_k} \alpha_k\right).
\]

(2) If \( n \) is even integers, then

\[
\prod_{i=1}^{n} \sin \alpha_i = \frac{(-1)^{[\frac{n}{2}]}}{2^{n-1}} \sum_{2 \leq k \leq n} \sum_{i_k=0}^{1} (-1)^{i_2+\cdots+i_n} \cos \left(\alpha_1 + \sum_{k=2}^{n} (-1)^{i_k} \alpha_k\right).
\]

Proof. First we assume that \( n \geq 1 \). We use induction on \( n \). If \( n = 1 \), then the formula (1) and if \( n = 2 \), then the formula (2) here are trivial. Assume, then, that the formula (2) in Lemma 1 holds true for \( n = 2, 4, \cdots, 2l \), i.e.,

\[
\prod_{i=1}^{2l} \sin \alpha_i = \frac{(-1)^l}{2^{2l-1}} \sum_{2 \leq k \leq 2l} \sum_{i_k=0}^{1} (-1)^{i_2+\cdots+i_{2l}} \cos \left(\alpha_1 + \sum_{k=2}^{2l} (-1)^{i_k} \alpha_k\right).
\]

We must show that the formula (2) in Lemma 1 is true when \( n = 2l + 2 \).
Set $n = 2l + 2$. We have

\[
\prod_{i=1}^{2l+2} \sin \alpha_i = \left\{ \frac{(-1)^l}{2^{2l-1}} \sum_{2 \leq k \leq 2l} \sum_{i_k = 0}^{1} (-1)^{i_2 + \cdots + i_{2l}} \cos \left( \alpha_1 + \sum_{k=2}^{2l+2} (-1)^{i_k} \alpha_k \right) \right\} \times \sin \alpha_{2l+1} \sin \alpha_{2l+2} \\
= \frac{(-1)^{l+1}}{2^{2l+1}} \sum_{2 \leq k \leq 2l+2} \sum_{i_k = 0}^{1} (-1)^{i_2 + \cdots + i_{2l+2}} \cos \left( \alpha_1 + \sum_{k=2}^{2l+2} (-1)^{i_k} \alpha_k \right).
\]

Thus, the formula (2) is true for $n = 2l + 2$. Similarly we can prove the formula (1) in Lemma 1.

Now let’s look at the product formula for the sine function:

\[
\prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) = \frac{\sin \pi z}{\pi z}, \quad z \in \mathbb{C}.
\]

For $s \in \mathbb{N}$ and $z \in \mathbb{C}$, it is well known [1] that

\[
(2.1) \quad \prod_{n=1}^{\infty} \left( 1 - \frac{z^s}{n^s} \right) = \sum_{n=0}^{\infty} (-1)^n \zeta(\{s\}_n) z^{ns}.
\]

Let $s = 4l + 2$ in (2.1). We obtain

\[
\sum_{n=0}^{\infty} (-1)^n \zeta(\{4l + 2\}_n) z^{(4l+2)n} = \prod_{n=1}^{\infty} \left( 1 - \frac{z^{4l+2}}{n^{4l+2}} \right) = \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) \left( 1 - \omega^2_{2l+1} \frac{z^2}{n^2} \right) \cdots \left( 1 - \omega^4_{2l+1} \frac{z^2}{n^2} \right) = \frac{1}{(\pi z)^{2l+1}} \prod_{i=1}^{2l+1} \sin(\pi \omega_{2l+1}^i z),
\]

\]

\]

\]
where \( \omega_l = \cos(2\pi/l) + i\sin(2\pi/l) \) is the \( l \)-primitive root of unity.

By Lemma 1, we have

\[
\sum_{n=0}^{\infty} (-1)^n \zeta(\{4l + 2\}_n) z^{(4l+2)n} = \frac{(-1)^l}{2^{2l}(\pi z)^{2l+1}} \sum_{1 \leq k \leq 2l} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l}} \sin \left\{ \left( 1 + \sum_{k=1}^{2l} (-1)^{i_k} \omega_{2l+1}^k \right) \pi z \right\}
\]

\[
= \frac{(-1)^l}{2^{2l}(\pi z)^{2l+1}} \sum_{1 \leq k \leq 2l} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l}} \times \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m+1)!} \left( 1 + \sum_{k=1}^{2l} (-1)^{i_k} \omega_{2l+1}^k \right)^{2m+1} (\pi z)^{2m+1}
\]

\[
= \sum_{m=0}^{\infty} \frac{(-1)^{m+l}(\pi z)^{2m-2l}}{2^{2l}(2m+1)!} \times \sum_{1 \leq k \leq 2l} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l}} \left( 1 + \sum_{k=1}^{2l} (-1)^{i_k} \omega_{2l+1}^k \right)^{2m+1}.
\]

Therefore, we have the formula

\[
\sum_{n=0}^{\infty} (-1)^n \zeta(\{4l + 2\}_n) z^{(4l+2)n} = \sum_{m=0}^{\infty} \frac{(-1)^{m+l}(\pi z)^{2m-2l}}{2^{2l}(2m+1)!} \times \sum_{1 \leq k \leq 2l} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l}} \left( 1 + \sum_{k=1}^{2l} (-1)^{i_k} \omega_{2l+1}^k \right)^{2m+1}.
\]

Let us compare the coefficients of the both sides of the above formula. Then we have

\[
\zeta(\{4l + 2\}_n) \quad 2^{-2l} \pi^{(4l+2)n} (2l+1)(2n+1)!
\]

\[
\times \sum_{1 \leq k \leq 2l} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l}} \left( 1 + \sum_{k=1}^{2l} (-1)^{i_k} \omega_{2l+1}^k \right)^{(2l+1)(2n+1)},
\]

(2.2)
since \((4l + 2)n = 2m - 2l\) for \(n \in \mathbb{N}\).

If \((4l + 2)n \neq 2m - 2l\) for \(n \in \mathbb{N}\), then we have

\[
(2.3) \quad \sum_{1 \leq k \leq 2l} \sum_{i_k=0}^{1} (-1)^{i_1 + \cdots + i_{2l-1}} \left( 1 + \sum_{k=1}^{2l} (-1)^{i_k} \omega_{2l+1}^k \right)^{2m+1} = 0.
\]

Let \(s = 4l\) in (2.1). We obtain

\[
\sum_{n=0}^{\infty} (-1)^n \zeta (\{4l\}_n) z^{4ln} = \prod_{n=1}^{\infty} \left( 1 - \frac{z^{4l}}{n^{4l}} \right) = \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right) \left( 1 - \omega_{4l}^2 \frac{z^2}{n^2} \right) \cdots \left( 1 - \omega_{4l}^{4l-2} \frac{z^2}{n^2} \right)
= \frac{(-1)^{l+1}}{(\pi z)^{2l}} \prod_{i=0}^{2l-1} \sin (\pi \omega_{4l}^i z).
\]

By Lemma 1, we have

\[
\sum_{n=0}^{\infty} (-1)^n \zeta (\{4l\}_n) z^{4ln} = \frac{2i}{(2\pi z)^{2l}} \sum_{1 \leq k \leq 2l-1} \sum_{i_k=0}^{1} (-1)^{i_1 + \cdots + i_{2l-1}} \cos \left\{ \left( 1 + \sum_{k=1}^{2l-1} (-1)^{i_k} \omega_{4l}^k \right) \pi z \right\}
= \frac{2i}{(2\pi z)^{2l}} \sum_{1 \leq k \leq 2l-1} \sum_{i_k=0}^{1} (-1)^{i_1 + \cdots + i_{2l-1}}
\]

\[
\times \sum_{m=0}^{\infty} (-1)^m \frac{(\pi z)^{2m}}{(2m)!} \left( 1 + \sum_{k=1}^{2l-1} (-1)^{i_k} \omega_{4l}^k \right)^{2m}
= \sum_{m=0}^{\infty} (-1)^m \frac{(\pi z)^{2m-2l}}{2^{2l-1}(2m)!} \prod_{i=0}^{2l-1} \sin (\pi \omega_{4l}^i z)
\]

\[
\times \sum_{1 \leq k \leq 2l-1} \sum_{i_k=0}^{1} (-1)^{i_1 + \cdots + i_{2l-1}} \left( 1 + \sum_{k=1}^{2l-1} (-1)^{i_k} \omega_{4l}^k \right)^{2m}.
\]
Therefore we have the formula:

\[
\sum_{n=0}^{\infty} (-1)^n \zeta(\{4l\}_n) z^{4ln} = \sum_{m=0}^{\infty} (-1)^m \frac{(\pi z)^{2m-2l} i}{2^{2l-1}(2m)!} \sum_{1 \leq k \leq 2l-1} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l-1}} \left( 1 + \sum_{k=1}^{2l-1} (-1)^{i_k} \omega_{4l}^k \right)^{2m}.
\]

Let us compare the coefficients of the both sides of the above formula. Then we have

\[
\zeta(\{4l\}_n) = (-1)^{n+l} \frac{2^{1-2l} \pi^{4ln} i}{(2l(2n+1))!} \sum_{1 \leq k \leq 2l-1} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l-1}} \left( 1 + \sum_{k=1}^{2l-1} (-1)^{i_k} \omega_{4l}^k \right)^{2l(2n+1)}.
\]

(2.4)

since \(4ln = 2m - 2l\) for \(n \in \mathbb{N}\).

Let \(4ln \neq 2m - 2l\) for \(n \in \mathbb{N}\), then we have

\[
\sum_{1 \leq k \leq 2l-1} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l-1}} \left( 1 + \sum_{k=1}^{2l-1} (-1)^{i_k} \omega_{4l}^k \right)^{2m} = 0.
\]

(2.5)

By (2.2) and (2.4), we have the following proposition.

**Proposition 1.** Let \(n \in \mathbb{N}\).

1. If \(l \in \mathbb{Z}_+\), then we have

\[
\zeta(\{4l+2\}_n) = \frac{2^{-2l} \pi^{(4l+2)n}}{((2n+1)(2l+1))!} \sum_{1 \leq k \leq 2l} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l}} \times \left( 1 + \sum_{k=1}^{2l} (-1)^{i_k} \omega_{2l+1}^k \right)^{(2l+1)(2n+1)}.
\]
(2) If \( l \in \mathbb{N} \), then we have
\[
\zeta(\{4l\}_n) = (-1)^{n+l} \frac{2^{1-2l}n^{4ln_i}}{(2(2n+1))!} \sum_{1 \leq k \leq 2l-1} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l-1}} \\
\times \left( 1 + \sum_{k=1}^{2l-1} (-1)^{i_k} \omega_{4l}^k \right)^{2l(2n+1)}.
\]

By (2.3) and (2.5), we have the following proposition.

**Proposition 2.** Let \( \omega_l = \cos(2\pi/l) + i \sin(2\pi/l) \) be the \( l \)-th primitive roots of unity and \( n, l \in \mathbb{N} \), then we have

(1) If \( m \neq 2ln + n + l \), then
\[
\sum_{1 \leq k \leq 2l} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l}} \left( 1 + \sum_{k=1}^{2l} (-1)^{i_k} \omega_{2l+1}^k \right)^{2m+1} = 0.
\]

(2) If \( m \neq 2ln + l \), then
\[
\sum_{1 \leq k \leq 2l-1} \sum_{i_k=0}^{1} (-1)^{i_1+\cdots+i_{2l-1}} \left( 1 + \sum_{k=1}^{2l-1} (-1)^{i_k} \omega_{4l}^k \right)^{2m} = 0.
\]

Now, we consider the multiple zeta values at even integers \( s = 2l \) in (2.1) to relate the Bernoulli numbers and Genocchi numbers.

Put
\[
g(z) = \sum_{n=1}^{\infty} \frac{B_{2n}(2\pi i)^{2n}}{2n(2n)!} z^{2n}.
\]

and \( f(z) = \log \frac{\sin \pi z}{\pi z} \). We have
\[
f'(z) = \frac{d}{dz} \log \left( \frac{\sin \pi z}{\pi z} \right) = \frac{d}{dz} \log \left( \frac{e^{\pi iz} - e^{-\pi iz}}{2\pi iz} \right)
\]
\[
= \frac{2\pi i}{e^{2\pi iz} - 1} + \pi i - \frac{1}{z}
\]
\[
= \sum_{n=0}^{\infty} \frac{B_n(2\pi i)^n}{n!} z^{n-1} + \pi i + \frac{1}{z}
\]
\[
= \sum_{n=2}^{\infty} \frac{B_n(2\pi i)^n}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{B_{2n}(2\pi i)^{2n}}{(2n)!} z^{2n-1}
\]
\[
= g'(z).
\]
Hence $f(z) = g(z)$ since $f(0) = g(0)$.
Let $s = 4l + 2$ in (2.1). We have

$$
\sum_{n=0}^{\infty} (-1)^n \zeta(\{4l + 2\}_n) z^{(4l+2)n}
= \frac{(-1)^l}{(2\pi z)^{2l}} \sum_{1 \leq k \leq 2l \ i_k = 0} \sum_{1 \leq k \leq 2l \ i_k = 0} (-1)^{i_1 + \cdots + i_{2l}} \sin((1 + \sum_{k=1}^{2l} (-1)^k \omega_{2l+1}^k) \pi z) \pi z
$$

$$
= \frac{(-1)^l}{(2\pi z)^{2l}} \sum_{1 \leq k \leq 2l \ i_k = 0} \sum_{1 \leq k \leq 2l \ i_k = 0} (-1)^{i_1 + \cdots + i_{2l}} \left(1 + \sum_{k=1}^{2l} (-1)^i_k \omega_{2l+1}^k \right)
\times \exp \left[ g \left\{ \left(1 + \sum_{k=1}^{2l} (-1)^i_k \omega_{2l+1}^k \right) z \right\} \right]
$$

$$
= \frac{(-1)^l}{(2\pi z)^{2l}} \sum_{1 \leq k \leq 2l \ i_k = 0} \sum_{1 \leq k \leq 2l \ i_k = 0} (-1)^{i_1 + \cdots + i_{2l}} \left(1 + \sum_{k=1}^{2l} (-1)^i_k \omega_{2l+1}^k \right)
\times \sum_{m=0}^{\infty} \frac{1}{m!} \left[ g \left\{ \left(1 + \sum_{k=1}^{2l} (-1)^i_k \omega_{2l+1}^k \right) z \right\} \right]^m.
$$

By the multinomial theorem we obtain

$$
\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{a_1 + \cdots + a_{2n+l+n} = m} \left( a_1, \cdots, a_{2n+l+n} \right) \left( a_1, \cdots, a_{2n+l+n} \right)
\times \prod_{j=1}^{2ln+l+n} \left[ B_{2j} \left\{ \left(1 + \sum_{k=1}^{2l} (-1)^i_k \omega_{2l+1}^k \right) 2\pi iz \right\}^{2j} \right] \frac{a_j}{2j(2j)!}
$$

$$
= \sum_{2ln+l+n=0}^{\infty} \left\{ \left(1 + \sum_{k=1}^{2l} (-1)^i_k \omega_{2l+1}^k \right) 2\pi iz \right\}^{4ln+2l+2n}
\times \sum_{(a_1, \cdots, a_{2n+l+n}) \in S(\alpha; 2n+l+n)} \prod_{j=1}^{2ln+l+n} \frac{1}{a_j!} \left( \frac{B_{2j}}{2j(2j)!} \right)^{a_j}.
$$

Comparing the coefficients of $z^{(4l+2)n}$ in the above formulae and using the Proposition 2, the proof of Theorem 1 is complete.
Put
\[
\tilde{g}(z) = - \sum_{n=1}^{\infty} \frac{G_{2n}(2\pi i)^{2n}}{4n(2n)!} z^{2n},
\]
and \(\tilde{f}(z) = \log \cos \pi z\). We have

\[
\tilde{f}'(z) = \frac{d}{dz} \log \cos \pi z = \frac{d}{dz} \log \left( \frac{e^{\pi iz} + e^{-\pi iz}}{2} \right)
= \pi i - \frac{2\pi i}{e^{2\pi iz} + 1}
= - \frac{1}{2} \sum_{n=1}^{\infty} \frac{G_{2n}(2\pi i)^{2n}}{(2n)!} z^{2n-1}
= \tilde{g}'(z).
\]

Hence \(\tilde{f}(z) = \tilde{g}(z)\) since \(\tilde{f}(0) = \tilde{g}(0)\).

Let \(s = 4l\) in (2.1). We have

\[
\sum_{n=0}^{\infty} (-1)^n \zeta(\{4l\}_n) z^{4ln}
= \frac{2i}{(2\pi z)^{2l}} \sum_{1 \leq k \leq 2l-1} \sum_{i_k = 0}^{1} (-1)^{i_1 + \cdots + i_{2l-1}} \cos \left( \left( 1 + \sum_{k=1}^{2l-1} (-1)^{i_k} \omega_{4l}^k \right) \pi z \right) \\
= \frac{2i}{(2\pi z)^{2l}} \sum_{1 \leq k \leq 2l-1} \sum_{i_k = 0}^{1} (-1)^{i_1 + \cdots + i_{2l-1}} \\
\times \exp \left[ \tilde{g} \left( \left( 1 + \sum_{k=1}^{2l-1} (-1)^{i_k} \omega_{4l}^k \right) z \right) \right] \\
= \frac{2i}{(2\pi z)^{2l}} \sum_{1 \leq k \leq 2l-1} \sum_{i_k = 0}^{1} (-1)^{i_1 + \cdots + i_{2l-1}} \\
\times \sum_{m=0}^{\infty} \frac{1}{m!} \left[ \tilde{g} \left( \left( 1 + \sum_{k=1}^{2l-1} (-1)^{i_k} \omega_{4l}^k \right) z \right) \right]^m.
\]
By the multinomial theorem we obtain
\[
\sum_{m=0}^{\infty} \frac{1}{m!} \sum_{a_1 + \cdots + a_{2l+n+l} = m} \binom{m}{a_1, \cdots, a_{2l+n+l}}
\times \prod_{j=1}^{2l+n+l} \left[ \frac{G_{2j} \left\{ (1 + \sum_{k=1}^{2l-1} (-1)^{ik} \omega_{4l}^k) 2\pi i z \right\}^{2j}}{4j(2j)!} \right]^{a_j}
= \sum_{2l+n+l=0}^{\infty} \left\{ \left( 1 + \sum_{k=1}^{2l-1} (-1)^{ik} \omega_{4l}^k \right) 2\pi i z \right\}^{4l+2l}
\times \sum_{(a_1, \cdots, a_{2l+n+l}) \in S(a; 2l+n+l)} \prod_{j=1}^{2l+n+l} \frac{1}{a_j!} \left( -\frac{G_{2j}}{4j(2j)!} \right)^{a_j}.
\]

Comparing the coefficients of \( z^{(4l+2)n} \) in the above formulae and using the Proposition 2, the proof of Theorem 2 is complete.

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