REMARKS ON SOME COMBINATORIAL DETERMINANTS

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Abstract. In this note we first give a simple, direct proof of a combinatorial determinant involving the usual higher derivatives and then obtain a corresponding result in positive characteristic.

1. Introduction

Let \( k[[x]] \) be the ring of formal power series in a variable \( x \) over the underlying field \( k \). Over a field \( k \) of characteristic 0, the following combinatorial determinant is known in [8]. Indeed, it concerns an evaluation of the Wronskian of powers of a single power series.

**Theorem A.** For \( f \in k[[x]] \), we have

\[
\det \left\{ \left( \frac{d^j}{dx^j} f(x)^i \right)_{i,j=0}^n \right\} = 1!^2 \ldots n! f'(x)^n (n+1)/2 \quad (n = 0, 1, 2, \ldots).
\]

This result follows from a property of the Wronskian:

\[
W_x (f_0(y), f_1(y), \ldots, f_n(y))
= (dy/dx)^{(n+1)/2} W_y (f_0(y), f_1(y), \ldots, f_n(y)),
\]

where \( W_x (f_0(y), f_1(y), \ldots, f_n(y)) \) is defined as determinant of \((n + 1) \times (n + 1)\) matrix whose \( i\)-th row is the vector \( \left( \frac{d^i}{dx^i} f_0(y), \frac{d^i}{dx^i} f_1(y), \ldots, \frac{d^i}{dx^i} f_n(y) \right) \) and \( f_i(0 \leq i \leq n) \) are at least \( n \) times differentiable functions. In what follows we shall deal with formal differentiability of a single function in \( k[[x]] \) without mentioning convergence of it.

The purpose of this note is to first give a simple, direct proof of Theorem A and then to establish its analogue in positive characteristic. Besides, we derive an amusing identity among the coefficients of a twisted formal power series in the Frobenius endomorphism \( \tau \). For Theorem A leads to a determinantal identity among the coefficients of a formal power series.

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2. Preliminaries

As it stands, the formula for \( n \geq p \) in Theorem A is not interesting in the case where the underlying field \( k \) is of prime characteristic \( p \). In order to avoid this case we will consider the Hasse-Teichmüller operators \( D^{(j)} \), which are defined by

\[
D^{(j)} \left( \sum_{i=0}^{\infty} a_i x^i \right) = \sum_{i=0}^{\infty} \binom{i}{j} a_i x^{i-j}
\]
on the ring \( k[[x]] \) of formal power series with coefficients in a field \( k \) of arbitrary characteristic. When the field \( k \) has characteristic 0, \( D^{(j)} \) are closely related by the usual higher derivatives \( \frac{d^j}{dx^j} \):

\[
D^{(j)} = \frac{1}{j!} \frac{d^j}{dx^j}.
\]

Even when the field \( k \) has positive characteristic \( p \), it is easily seen that \( D^{(j)} \) is a non-trivial operator for each integer \( j \geq p \). Like the ordinary higher derivatives, the Hasse-Teichmüller operators satisfy the product rule [10], quotient rule [2] and chain rule [3, 4]. We refer to [5] for a concise summary on all of these properties.

3. Equivalent statement

It is evident from the relation (1) that Theorem A is equivalent to the following statement.

**Theorem B.** For \( f \in k[[x]] \), we have

\[
\det \left\{ \left( D^{(j)} f(x)^i \right)^n \right\}_{i,j=0} = f'(x)^{n(n+1)/2} \quad (n = 0, 1, 2, \ldots).
\]

To give a proof of Theorem B, we here state a special case of the product rule for \( D^{(j)} \) what we call, the power rule (see [10, 1]).

**Power rule.** For all \( f \in k[[x]] \) and \( j > 0 \),

\[
D^{(j)} (f^e) = \sum_{e=1}^{j} \sum_{\lambda_1 + \lambda_2 + \cdots + \lambda_e = j, \lambda_i > 0} \binom{i}{e} f^{i-e} D^{(\lambda_1)} f \cdot D^{(\lambda_2)} f \cdots D^{(\lambda_e)} f.
\]

**Proof of Theorem B.** Put a matrix \( b = (b_{i,j}) \) by

\[
b_{i,j} = (-1)^{i+j} \binom{i}{j} f^{i-j} \quad (i, j \geq 0)
\]

and a matrix \( c = (c_{i,j}) \) by \( c_{i,j} = D^{(j)} f^i \). Then the theorem will follow from the fact that \( bc \) is an upper triangular matrix with diagonal entries \( (bc)_{i,i} = (D^{(1)} f)^i = f^{i^2} \). Now we use the power rule to compute \( (bc)_{i,j} \) for \( 0 \leq j \leq i \leq n \) as follows:
\[(bc)_{i,j} = \sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} f^{i-k} D^{(j)} f^k\]

\[= \sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} f^{i-k} \sum_{\lambda_1 + \lambda_2 + \cdots + \lambda_s = j, \lambda_1 > 0} \binom{k}{\lambda_1} f^{k-\lambda_1} D^{(\lambda_1)} f \cdot D^{(\lambda_2)} f \cdots D^{(\lambda_s)} (f)\]

\[= \sum_{k=0}^{i} \sum_{\lambda_1 + \lambda_2 + \cdots + \lambda_s = j, \lambda_1 > 0} (-1)^{i+k} \binom{i}{k} \binom{k}{\lambda_1} f^{i-\lambda_1} D^{(\lambda_1)} f \cdot D^{(\lambda_2)} f \cdots D^{(\lambda_s)} (f)\]

\[= \sum_{\lambda_1 + \lambda_2 + \cdots + \lambda_s = j, \lambda_1 > 0} \binom{i}{\lambda_1} (\sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} \binom{i-\lambda_1}{i-k}) f^{i-\lambda_1} D^{(\lambda_1)} f \cdot D^{(\lambda_2)} f \cdots D^{(\lambda_s)} (f)\]

\[= \sum_{\lambda_1 + \lambda_2 + \cdots + \lambda_s = j, \lambda_1 > 0} \binom{i}{\lambda_1} (1 - (-1)^{i-\lambda_1} f^{i-\lambda_1} D^{(\lambda_1)} f) \cdot D^{(\lambda_2)} f \cdots D^{(\lambda_s)} (f).\]

The result follows now from the very last equality that \((bc)_{i,j} = 0\) for \(i > j\) and \((bc)_{i,i} = f^i\).

\[\square\]

As a special case, we immediately have the following corollary, which was first observed in [11]. We denote by \([x^j] f^i\) the coefficient of \(x^j\) in \(f^i\). Then it is easy to check that \(D^{(j)} f^i|_{x=0} = [x^j] f^i\).

**Corollary 1.** Let \(f(x) = 1 + a_1 x + a_2 x^2 + \cdots\) be a formal power series and define a matrix \(c\) by

\[c_{i,j} = [x^j] f^i\quad (i, j \geq 0).\]

Then

\[\det((c_{i,j})_{i,j=0}^n) = a_1^{n(n+1)/2}\quad (n = 0, 1, 2, \ldots).\]

As an application, we recall the Bernoulli numbers \(B_j^{(i)}\) of higher order \(i\), defined by

\[(\frac{x}{e^x - 1})^i = \sum_{j=0}^{\infty} \frac{B_j^{(i)}}{j!} x^j.\]

The following corollary is then immediate from \(B_1^{(1)} = B_1 = -\frac{1}{2}\).

**Corollary 2.**

\[\det((B_j^{(i)})_{i,j=0}^n) = (-1/2)^{n(n+1)/2} 1! \cdots n!\quad (n = 0, 1, 2, \ldots).\]

We now introduce another combinatorial determinant involving the operator \((x \frac{d}{dx})^j\), which is less well known but has a number of applications to series in
calculus. The operator has a nice expansion formula in terms of the usual higher derivatives (see [7]).

\[ (x \frac{d}{dx})^j f(x) = \sum_{e=1}^{j} S(j, e) x^e \frac{d^e f}{dx^e}, \]

where \( S(j, e) \) are Stirling numbers of the second kind. Together with the power rule and (1) we are able to use (2) to derive Theorem C analogous to Theorem A, in the same way we did in the proof of Theorem B.

**Theorem C.**

\[ \det \left\{ \left( (x \frac{d}{dx})^i f(x)^j \right)^{n}_{i,j=0} \right\} = n! \cdots 2! \cdots 1! \cdot (n f'(x))^{n+1}/2 \quad (n = 0, 1, 2, \ldots). \]

Along this work, there is another combinatorial determinant in the case where multiplication is replaced by composition.

**Theorem D.** Let \( f(x) = x + a_1 x^2 + a_2 x^3 + \cdots \) be a formal power series and define \( f^{(0)} = x \) and \( f^{(i)} = f(f^{(i-1)}) \) for \( i > 1 \). Define a matrix \( c \) by

\[ c_{i,j} = [x^{i+1}] f^{(i)} \quad (i, j \geq 0). \]

Then

\[ \det((c_{i,j})^{n}_{i,j=0}) = n! \cdots 2! \cdot 1! a_1^{n+1}/2 \quad (n = 0, 1, 2, \ldots). \]

**Proof.** See [6] \( \square \)

4. **An analogue in positive characteristic**

We are now in a position of obtaining the analogue of Theorem D in characteristic \( p > 0 \). To this end, let \( K \) denote a field of prime characteristic \( p \) in which a subfield \( \mathbb{F}_q \) of order \( q = p^m \) is imbedded, and let \( \tau \) be the Frobenius endomorphism on \( G_a/K \), that is, \( \tau(x) = x^q \). Let \( K\{\{\tau\}\} \) be the ring of formal power series in \( \tau \). Then multiplication in \( K\{\{\tau\}\} \) is twisted by the commutation relation

\[ \tau \alpha = \alpha^q \tau, \quad \alpha \in K. \]

More precisely, multiplication of two formal power series \( f(\tau) = \sum_{i=0}^{\infty} a_i \tau^i \), \( g(\tau) = \sum_{k=0}^{\infty} b_i \tau^i \) in \( K\{\{\tau\}\} \) is given by

\[ f(\tau) g(\tau) = \sum_{i=0}^{\infty} c_k \tau^k \quad \text{with} \quad c_k = \sum_{i+j=k} a_i b_j^q. \]

Let \( A_{\mathbb{F}_q}(K) \) denote the set of \( \mathbb{F}_q \)-additive formal power series with coefficients in \( K \). Then it is well known in [9] that \( A_{\mathbb{F}_q}(K) \) is a ring with standard addition and multiplication being given by composition and that the map \( K\{\{\tau\}\} \to A_{\mathbb{F}_q}(K) \), \( f(\tau) \mapsto f(\tau)(x) \) is isomorphic. We state the analogue of Theorem D for \( K\{\{\tau\}\} \).
**Theorem E.** Let \( f(\tau) = \tau^0 + a_1 \tau + a_2 \tau^2 + \cdots \) be a twisted formal power series in \( K\{\{\tau\}\} \) and define a matrix \( c \) by

\[
c_{i,j} = [\tau^j] f^i \quad (i, j \geq 0).
\]

Then

\[
\det((c_{i,j})_{i,j=0}^n) = a_1^{e(n)} \quad (n = 0, 1, 2, \ldots),
\]

where

\[
e(n) = \frac{1}{q-1} (\frac{q^n+1}{q-1} - (n+1)).
\]

By the isomorphism above, Theorem E is equivalent to the following statement.

**Theorem F.** Let \( f(\tau)(x) = x + a_1 x^q + a_2 x^{q^2} + \cdots \) be an \( \mathbb{F}_q \)-additive formal power series in \( A_{\mathbb{F}_q}(K) \) and define

\[
f^{(0)}(\tau)(x) = x \quad \text{and} \quad f^{(i)}(\tau)(x) = f(f^{(i-1)}(\tau)(x)) \]

for \( i > 1 \). Define a matrix \( c \) by

\[
c_{i,j} = [x^{q^j}] f^{(i)}(\tau)(x) \quad (i, j \geq 0).
\]

Then

\[
\det((c_{i,j})_{i,j=0}^n) = a_1^{e(n)}, \quad (n = 0, 1, 2, \ldots)
\]

where \( e(n) \) is as in Theorem E.

**Proof.** We show that the matrix

\[
b_{i,j} = (-1)^{i+j} \binom{i}{j} \quad (i, j \geq 0)
\]

has the property that \( bc \) is an upper triangular matrix. Put

\[
g_i(x) = \sum_k (-1)^{i+k} \binom{i}{k} f^{(k)}(x).
\]

Then \( (bc)_{i,j} = [x^{q^j}] g_i(x) \), and the theorem follows from the fact that

\[
g_i(x) = \sum_{j=0} b_{i,j}^{(i)} x^{q^j} \quad \text{with} \quad b_{0}^{(i)} = a_1^{1+q+\cdots+q^{i-1}}.
\]
which we prove by induction on $i$. For the case $i = 0$ it is obviously true. Assume that $g_{i-1}(x) = \sum_{j=0}^{\infty} b_j^{(i-1)} x^{q^{i+j-1}}$. Then we compute
\[
g_i(x) = g_{i-1}(f(x)) - g_{i-1}(x) = \sum_{j=0}^{\infty} b_j^{(i-1)} a_k^{i+j-1} \cdot x^{q^{i+j+k-1}} - \sum_{j=0}^{\infty} b_j^{(i-1)} x^{q^{i+j-1}}
\]
where $a_0 = 1$
\[
= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} b_j^{(i-1)} a_k^{i+j-1} x^{q^{i+j+k-1}} - \sum_{j=0}^{\infty} b_j^{(i-1)} x^{q^{i+j-1}}
\]
\[
= \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} b_j^{(i-1)} a_k^{i+j-1} x^{q^{i+j+k-1}}.
\]
Hence $b_0^{(i)} = b_0^{(i-1)} a_1^{i-1} = a_1^{1+q+\cdots+q^{i-1}}$, as desired. □

References