NONEXISTENCE OF RICCI-PARALLEL REAL HYPERSURFACES IN $P_2\mathbb{C}$ OR $H_2\mathbb{C}$

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ABSTRACT. Niebergall and Ryan posed many open problems on real hypersurfaces in complex space forms. One of them is “Are there any Ricci-parallel real hypersurfaces in complex projective space $P_2\mathbb{C}$ or complex hyperbolic space $H_2\mathbb{C}$?” The purpose of present paper is to prove the nonexistence of such hypersurfaces.

1. Introduction

A complex 2-dimensional Kaehlerian manifold of constant holomorphic sectional curvature $4c$ is called a complex space form, which is denoted by $M_2(c)$. A complete and simply connected complex space form consists of a complex projective space $P_2\mathbb{C}$, a complex Euclidean space $\mathbb{C}^2$ or a complex hyperbolic space $H_2\mathbb{C}$, according to $c > 0$, $c = 0$ or $c < 0$.

In [2], R. Niebergall and P. J. Ryan gave the necessary background material to access the study of real hypersurfaces in complex space forms and gave a survey of this field of the study. Also they posed many open problems. One of them is the following:

“Are there any Ricci-parallel real hypersurfaces in $P_2\mathbb{C}$ or $H_2\mathbb{C}$?”

The purpose of the present paper is to give a negative answer for this open problem.

2. Preliminaries

Let $(M_2(c), <, >, J)$ be a complex space form with constant holomorphic sectional curvature $4c(\neq 0)$ and with Levi-Civita connection $\nabla$. Let $M$ be a real hypersurface immersed in $M_2(c)$. Then, denoting

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the Riemannian metric on $M$ induced from the metric on $M_2(c)$ by the same symbol $\langle, \rangle$, the Levi-Civita connection $\nabla$ of the induced metric $\langle, \rangle$ and the shape operator $A$ of the immersion are characterized respectively by

$$\tilde{\nabla}_XY = \nabla_XY + \langle AX, Y \rangle \xi, \quad \tilde{\nabla}_X \xi = -AX,$$

where $\xi$ is a local choice of unit normal. Define the structure vector $W = -J\xi$. Then $W \in TM$ and $\langle W, W \rangle = 1$. Denote $\alpha = \langle AW, W \rangle$.

Define a skew-symmetric $(1, 1)$-tensor $\phi$ from the tangential projection of $J$ by

$$JX = \phi X + \langle X, W \rangle \xi.$$

Then we have

$$\phi^2 X = -X + \langle X, W \rangle W, \quad \phi W = 0,$$

$$\langle \phi X, \phi Y \rangle = \langle X, Y \rangle - \langle X, W \rangle \langle Y, W \rangle,$$

that is, $(\phi, W, \langle, \rangle)$ determines an almost contact metric structure ([1]).

The Gauss and Codazzi equations are given by

$$(2.2) \quad R(X, Y)Z = c[\langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle \phi Y, Z \rangle \phi X - \langle \phi X, Z \rangle \phi Y$$

$$- 2 \langle \phi X, Y \rangle \phi Z] + \langle AY, Z \rangle AX - \langle AX, Z \rangle AY,$$

$$(2.3) \quad (\nabla_X A)Y - (\nabla_Y A)X$$

$$= c(\langle X, W \rangle \phi Y - \langle Y, W \rangle \phi X + 2 \langle X, \phi Y \rangle W).$$

From equation (2.2) we get the Ricci tensor $S$ of type $(1, 1)$ as

$$(2.4) \quad SX = 5cX - 3c < X, W > W + mAX - A^2 X,$$

where $m = \text{trace} A$ is the mean curvature of $M$.

It is well known ([2]) that

$$(2.5) \quad \nabla_X W = \phi AX,$$

$$(2.6) \quad (\nabla_X \phi)Y = \langle Y, W \rangle AX - \langle AX, Y \rangle W.$$

If $W$ is a principal vector, then $M$ is called a Hopf hypersurface ([2]). In a 3-dimensional Hopf hypersurface, the present author proved the following

**Theorem 2.1** ([3]). Let $M$ be a 3-dimensional Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then $M$ cannot have harmonic curvature, that is, $(\nabla_Y S)Z - (\nabla_Z S)Y$ cannot vanish identically.
3. Nonexistence of Ricci-parallel real hypersurfaces in complex space forms $P_2\mathbb{C}$ or $H_2\mathbb{C}$

Let $M$ be a 3-dimensional real hypersurface in complex space form $M_2(c)$ with constant holomorphic sectional curvature $4c \neq 0$. Suppose that $M$ is a Ricci-parallel real hypersurface. Then, by Theorem 2.1, $M$ is not a Hopf hypersurface. We choose a local orthonormal frame field $W, X, \phi X$ of $M$ and put

$$AW = \alpha W + bX + e\phi X,$$

$$AX = bW + \beta X + \delta \phi X,$$

$$A\phi X = eW + \delta X + \gamma \phi X,$$

where we have used the property $<AY, Z> = <Y, AZ>$. Since $M$ is not a Hopf hypersurface, the structure vector field $W$ is not principal at some point $P$ of $M$. Hence $b \neq 0$ in an open neighborhood $U_1$ of $P$ or $e \neq 0$ in an open neighborhood $U_2$ of $P$. Thus, $b \neq 0$ in an open neighborhood $U$ of $P$ or $e \neq 0$ in an open neighborhood $U$ of $P$. Hereafter we consider the local frame field $W, X, \phi X$ in $U$ only.

Case A. Assume that the function $\delta$ is always zero in $U$.

Then the equation (3.1) can be written as follows.

$$AW = \alpha W + bX + e\phi X,$$

$$AX = bW + \beta X,$$

$$A\phi X = eW + \gamma \phi X.$$

From (3.2), $m$ is given by $m = \alpha + \beta + \gamma$. From (2.4) and (3.2), we have

$$SW = (2c + \alpha\beta + \alpha\gamma - b^2 - e^2)W + b\gamma X + e\beta \phi X,$$

$$SX = b\gamma W + (5c + \alpha\beta + \beta\gamma - b^2)X - be \phi X,$$

$$S\phi X = e\beta W - beX + (5c + \alpha\gamma + \beta\gamma - e^2)\phi X.$$

Since $<\nabla_W X, W> = -<X, \nabla_W W> = -<X, \phi AW> = -<X, b\phi X - eX> = e$, we can put

$$\nabla_W X = eW + f\phi X,$$

where we have put $f = <\nabla_W X, \phi X>$. Since $<\nabla_X X, W> = -<X, \nabla_X W> = -<X, \beta \phi X> = 0$, we can put

$$\nabla_X X = h\phi X,$$
where we have put \( h = \langle \nabla_X X, \phi X \rangle \). Similarly, we can put
\[
\nabla_{\phi X} X = \gamma W + i \phi X,
\]
where we have put \( i = \langle \nabla_{\phi X} X, \phi X \rangle \). From (3.4) and (3.5) we obtain, by the help of (2.6), (3.2), (3.4) and (3.5),
\[
\begin{align*}
\nabla_X \nabla_W X &= (Xe - f \beta)W - fhX + (e\beta + Xf)\phi X, \\
\nabla_W \nabla_X X &= -bhW - fhX + (Wh)\phi X, \\
\nabla_X W - \nabla_W X &= -eW + (\beta - f)\phi X.
\end{align*}
\]
Since
\[
\langle \nabla_X \nabla_W X - \nabla_W \nabla_X X - \nabla \nabla_W X - \nabla_W X, W \rangle = \langle R(X, W)X, W \rangle
\]
and
\[
\langle \nabla_X \nabla_W X - \nabla_W \nabla_X X - \nabla \nabla_W X - \nabla_W X, \phi X \rangle = \langle R(X, W)X, \phi X \rangle,
\]
we have from (2.2), (3.2), (3.4), and (3.6),
\[
\begin{align*}
Xe &= f\beta - bhX - (f - \beta)\gamma - e^2 - c + b^2 - \alpha\beta, \\
Wh - Xf &= 2e\beta + efX - (\beta - f)i.
\end{align*}
\]
Since \( M \) has the parallel Ricci tensor \( S \), we have from the first equation of (3.3)
\[
S \nabla_W W = W(\alpha\beta + \alpha\gamma - b^2 - e^2)W + (2c + \alpha\beta + \alpha\gamma - b^2 - e^2)\nabla_W W \\
+ W(b\gamma)X + b\gamma \nabla_W X + W(e\beta)\phi X \\
+ e\beta(\nabla_W \phi)X + e\beta\phi \nabla_W X,
\]
from which and (2.5) and (2.6) we get, by the help of (3.2) and (3.3),
\[
\begin{align*}
b\{e\beta W - beX + (5c + \alpha\gamma + \beta\gamma - e^2)\phi X\} \\
- e\{b\gamma W + (5c + \alpha\beta + \beta\gamma - b^2)X - be\phi X\} \\
= W(\alpha\beta + \alpha\gamma - b^2 - e^2)W + b(2c + \alpha\beta + \alpha\gamma - b^2 - e^2)\phi X \\
- e(2c + \alpha\beta + \alpha\gamma - b^2 - e^2)X \\
+ W(b\gamma)X + b\gamma \nabla_W X + W(e\beta)\phi X - be\beta W + e\beta\phi \nabla_W X.
\end{align*}
\]
Taking inner product (3.9) with \( W \), \( X \) and \( \phi X \), respectively, we have
\[
\begin{align*}
W(\alpha\beta + \alpha\gamma - b^2 - e^2) &= 2e\beta(\beta - \gamma), \\
W(b\gamma) &= e(-3c + \alpha\gamma - \beta\gamma - b^2 - e^2 + f\beta), \\
W(e\beta) &= b(3c - \alpha\beta + \beta\gamma + b^2 + e^2 - f\gamma).
\end{align*}
\]
Differentiating the first equation of (3.3) with respect to $X$ covariantly, we obtain, by the help of (2.5), (2.6), (3.2), (3.3) and (3.5),

\begin{align}
(3.13) & \quad X(\alpha\beta + \alpha\gamma - b^2 - e^2) = 2e\beta^2, \\
(3.14) & \quad X(b\gamma) = (h - b)e\beta, \\
(3.15) & \quad X(e\beta) = \beta(3c + \beta\gamma - \alpha\beta + b^2) - bh\gamma.
\end{align}

Differentiating the first equation of (3.3) with respect to $\phi X$ and taking account of (2.5), (2.6), (3.2), (3.3) and (3.6), we have

\begin{align}
(3.16) & \quad \phi X(\alpha\beta + \alpha\gamma - b^2 - e^2) = -2b\gamma^2, \\
(3.17) & \quad \phi X(b\gamma) = -\gamma(3c + \beta\gamma - \alpha\gamma + e^2) + ie\beta, \\
(3.18) & \quad \phi X(e\beta) = b\gamma(e - i).
\end{align}

Differentiating the second equation of (3.3) with respect to $W$, $X$ and $\phi X$, respectively and taking inner product the resulting equations with $X$ and $\phi X$, respectively, we obtain

\begin{align}
(3.19) & \quad W(\alpha\beta + \beta\gamma - b^2) = 2be(\gamma - f), \\
(3.20) & \quad W(be) = -f(\alpha\gamma - \alpha\beta + b^2 - e^2) - e^2\beta + b^2\gamma, \\
(3.21) & \quad X(\alpha\beta + \beta\gamma - b^2) = -2b\beta h, \\
(3.22) & \quad X(be) = -h(\alpha\gamma - \alpha\beta + b^2 - e^2) + b\beta\gamma, \\
(3.23) & \quad \phi X(\alpha\beta + \beta\gamma - b^2) = 2b\gamma^2 - 2bei, \\
(3.24) & \quad \phi X(be) = -i(\alpha\gamma - \alpha\beta + b^2 - e^2) - e\beta\gamma.
\end{align}

Differentiating the third equation of (3.3) with respect to $W$, $X$ and $\phi X$, respectively and taking inner product the resulting equations with respect to $\phi X$, we have

\begin{align}
(3.25) & \quad W(\alpha\gamma + \beta\gamma - e^2) = 2be(f - \beta), \\
(3.26) & \quad X(\alpha\gamma + \beta\gamma - e^2) = 2e(bh - \beta^2), \\
(3.27) & \quad \phi X(\alpha\gamma + \beta\gamma - e^2) = 2bei.
\end{align}

From (3.3), the scalar curvature $s$ of $M$ is given by

\begin{equation}
(3.28) \quad s = 12c + 2(\alpha\beta + \alpha\gamma + \beta\gamma - b^2 - e^2).
\end{equation}

Since the Ricci tensor $S$ is parallel, we have from (3.28)

\begin{equation}
(3.29) \quad Z(\alpha\beta + \alpha\gamma + \beta\gamma - b^2 - e^2) = 0
\end{equation}
for every vector field $Z$. Hence we have from (3.10), (3.13), (3.16), (3.19), (3.21), (3.23), (3.25), (3.26), (3.27) and (3.29)

\begin{align*}
(3.30) & \quad W(\beta \gamma) = 2be(\gamma - \beta), \\
(3.31) & \quad X(\beta \gamma) = -2e\beta^2, \\
(3.32) & \quad \phi X(\beta \gamma) = 2b\gamma^2, \\
(3.33) & \quad W(e^2 - \alpha \gamma) = 2be(\gamma - f), \\
(3.34) & \quad X(e^2 - \alpha \gamma) = -2beh, \\
(3.35) & \quad \phi X(e^2 - \alpha \gamma) = 2b(\gamma^2 - ei), \\
(3.36) & \quad W(b^2 - \alpha \beta) = 2be(f - \beta), \\
(3.37) & \quad X(b^2 - \alpha \beta) = 2e(bh - \beta^2), \\
(3.38) & \quad \phi X(b^2 - \alpha \beta) = 2bei.
\end{align*}

From $\nabla_X \nabla_W W - \nabla_W \nabla_X W - \nabla_{\nabla_X W - \nabla_W X} W = R(X, W)W$, we have

\[(3.39) \quad Xb - W\beta = -2be + eh.\]

Since $\{[X, W] - (\nabla_X W - \nabla_W X)\}(\beta \gamma) = 0$, we have from (2.5), (3.2), (3.4), (3.30), (3.31) and (3.32)

\[
(W\beta - Xb)e\beta - bX(e\beta) + \beta W(e\beta) + b\gamma X(e) + eX(b\gamma) - (\beta - f)b\gamma^2 + e(-be\beta + be\gamma) = 0.
\]

Substituting (3.7), (3.12), (3.14), (3.15) and (3.39) into the above equation, we find

\[(3.40) \quad be^2\beta = (c - b^2 + \alpha \beta)b\gamma.\]

From $\nabla_{\phi X} \nabla_W X - \nabla_W \nabla_{\phi X} X - \nabla_{\nabla_{\phi X} W - \nabla_W \phi X} X = R(\phi X, W)X$, we obtain

\[
(3.41) \quad (\phi X)e - W\gamma = b(2e - i),
\]

\[
(3.42) \quad (\phi X)f - Wi = 2b\gamma + bf + h(f - \gamma).
\]

From $\nabla_{\phi X} \nabla_W W - \nabla_W \nabla_{\phi X} W - \nabla_{\nabla_{\phi X} W - \nabla_W (\phi X)} W = R(\phi X, W)W$, we obtain

\[
(3.43) \quad (\phi X)b = ei - f\gamma + b^2 + \beta(f - \gamma) + \alpha \gamma - e^2 + c.
\]

Since $\{[\phi X, W] - (\nabla_{\phi X} W - \nabla_W (\phi X))\}(\beta \gamma) = 0$, we have

\[
-e\beta(\phi X)b - b(\phi X)(e\beta) + [(\phi X)e - W\gamma]b\gamma - \gamma W(b\gamma) + e(\phi X)(b\gamma)
+ (f - \gamma)e\beta^2 + b^2(e(\beta - \gamma) = 0.
\]
Substituting (3.11), (3.17), (3.18), (3.41) and (3.43) into this equation, we find

\[(3.44) \quad b^2e\gamma = e\beta(\alpha\gamma - e^2 + c).\]

From \(\nabla_{\phi X} \nabla_X X = \nabla_X \nabla_{\phi X} X - \nabla_{\nabla_X X} X - \nabla_X (\phi_X) X = R(\phi_X, X)X,\) we have

\[(3.45) \quad X\gamma = -2e\beta - (\gamma - \beta)i - e\gamma,\]
\[(3.46) \quad (\phi X)h - Xi = 4c + 2\beta\gamma + (\beta + \gamma)f + h^2 + i^2.\]

From \(\nabla_{\phi X} \nabla_X W = \nabla_X \nabla_{\phi X} W - \nabla_{\nabla_X X} X - \nabla_X (\phi_X) W = R(\phi_X, X)W,\) we have

\[(3.47) \quad (\phi X)\beta = (\beta - \gamma)h + b(\beta + 2\gamma).\]

From \(\{[\phi X, X] - (\nabla_{\phi X} X - \nabla_X (\phi_X))(e^2 - \alpha\gamma) = 0,\) we obtain, by the help of (3.33), (3.34) and (3.35),

\[-be\{(\phi X)h - Xi\} - h(\phi X)(be) - \gamma X(b\gamma) - b\gamma X(\gamma) + iX(be)\]
\[-be(\beta + \gamma)(\gamma - f) + beh^2 - bi(\gamma^2 - ie) = 0.\]

Substituting (3.14), (3.22), (3.24), (3.45) and (3.46) into this equation, we find

\[(3.48) \quad be = 0.\]

Now, we shall show that we have a contradiction in each case of \(b \neq 0\) and \(e \neq 0\) in \(\mathcal{U}.\)

Case 1. \(b \neq 0\) in \(\mathcal{U}.\)

In this case, we have \(e = 0\) in \(\mathcal{U}\) from (3.48).

Firstly, we shall show that

\[(3.49) \quad i = 0 \quad \text{in} \quad \mathcal{U}.\]

To show (3.49), assume that \(i \neq 0\) at a point \(Q \in \mathcal{U}.\) Since we have \(i(\alpha\gamma - \alpha\beta + b^2) = 0\) in \(\mathcal{U}\) from (3.24), we have \(\alpha\gamma - \alpha\beta + b^2 = 0\) in an open neighborhood \(\mathcal{U}' \subset \mathcal{U}\) of the point \(Q.\) Hence we have \(\gamma = 0\) in \(\mathcal{U}'\) from (3.20) and hence we get \(c = 0\) from (3.12) and \(b^2 - \alpha\beta = 0.\) This is absurd.

Secondly, we shall show that

\[(3.50) \quad b^2 - \alpha\beta = c \quad \text{in} \quad \mathcal{U}.\]

To show (3.50), assume that \(b^2 - \alpha\beta - c \neq 0\) at a point \(Q \in \mathcal{U}.\) Then we have \(\gamma = 0\) in an open neighborhood \(\mathcal{V} \subset \mathcal{U}\) of \(Q\) from (3.40). Since \(3c - \alpha\beta + b^2 = 0\) in \(\mathcal{V}\) from (3.12), we get \(f = 0\) in \(\mathcal{V}\) from (3.20). Hence
we get \( bh = -4c \) in \( \mathcal{V} \) from (3.7). Differentiating \( bh = -4c \) with respect to \( \phi X \) covariantly in \( \mathcal{V} \), we find, by the help of (3.43), (3.46) and (3.49),
\[
b(\phi X)h + h(\phi X)b = b(4c + h^2) + h(b^2 + c) = 4bc - 4ch - 4bc + ch = -3ch = 0,
\]
which shows that \( h = 0 \). So, we get \( c = -\frac{1}{4}bh = 0 \). This is absurd.

Thirdly, we shall show that
\[
(3.51) \quad h = 0 \quad \text{in} \quad \mathcal{U}.
\]
To show this, we start with the equation \( 4c + \beta \gamma = f \gamma \) from (3.12) and (3.50). Thus we get
\[
\gamma \phi X(\beta - f) + (\beta - f)\phi X(\gamma) = 0,
\]
which implies, by the help of (3.42), (3.47) and (3.49),
\[
(\beta - f)(\gamma b + \gamma h + \phi X(\gamma)) = 0 \quad \text{in} \quad \mathcal{U}.
\]
Since \( \beta \neq f \) from \( 4c + \beta \gamma = f \gamma \), we have \( (\phi X)\gamma = -\gamma(b + h) \). From (3.32), we have \( \gamma \phi X(\beta) + \beta \phi X(\gamma) = 2b\gamma^2 \). Hence we have \( h\gamma = 0 \) in \( \mathcal{U} \) from (3.47).

If \( h \neq 0 \) at a point \( Q \) in \( \mathcal{U} \), then we have \( \gamma = 0 \) in an open neighborhood \( \mathcal{V} \subset \mathcal{U} \) of \( Q \). Then we have \( 3c = \alpha \beta - b^2 \) from (3.12). This is impossible because of (3.50). Hence \( h = 0 \) in \( \mathcal{U} \).

From (3.22) and (3.51), we have \( \beta \gamma = 0 \) and hence \( \gamma = 0 \) in \( \mathcal{U} \) from (3.32). Thus we obtain \( 3c = \alpha \beta - b^2 \) from (3.12). The equations \( 3c = \alpha \beta - b^2 \) and (3.50) imply \( c = 0 \), which contradicts the hypothesis.

Therefore we have a contradiction in the case of \( b \neq 0 \) in an open set \( \mathcal{U} \) of \( M \).

Case 2. \( e \neq 0 \) in \( \mathcal{U} \).

In this case, we have \( b = 0 \) in \( \mathcal{U} \) from (3.48). From (3.4) (3.14), (3.11), we get in \( \mathcal{U} \) respectively
\[
(3.52) \beta(\alpha \gamma - e^2 + c) = 0,
\]
\[
(3.53) \quad h\beta = 0,
\]
\[
(3.54) \quad 3c + \beta \gamma = \alpha \gamma - e^2 + f\beta.
\]
From (3.22) and (3.53), we get
\[
(3.55) \quad h(\alpha \gamma - e^2) = 0.
\]
If \( h \neq 0 \) at a point \( Q \in \mathcal{U} \), then we have from (3.53) and (3.55)
\[
\beta = 0 \quad \text{and} \quad \alpha \gamma - e^2 = 0
\]
in an open neighborhood $\mathcal{V} \subset \mathcal{U}$ of $Q$. Hence we have $c = 0$ from (3.54). This is impossible. Thus we have in $\mathcal{U}$

\begin{equation}
(3.56) \quad h = 0.
\end{equation}

We shall show that

\begin{equation}
(3.57) \quad e^2 - \alpha \gamma - c = 0 \quad \text{in} \quad \mathcal{U}.
\end{equation}

To show this, assume that $e^2 - \alpha \gamma - c \neq 0$ at a point $Q$ in $\mathcal{U}$. Then we obtain $e^2 - \alpha \gamma - c \neq 0$ in open neighborhood $\mathcal{V} \subset \mathcal{U}$ of $Q$. Hence we get $\beta = 0$ in $\mathcal{V}$ from (3.52) and hence $\alpha \gamma - e^2 = 3c$ from (3.54). Moreover we get $f(\alpha \gamma - e^2) = 0$ and $i(\alpha \gamma - e^2) = 0$ in $\mathcal{V}$ from (3.20) and (3.24), respectively, which implies $f = i = 0$ in $\mathcal{V}$. Since $f = i = h = \beta = 0$ in $\mathcal{V}$, we have $c = 0$ from (3.46). This is absurd. Hence we get $e^2 - \alpha \gamma - c = 0$ in $\mathcal{U}$.

From (3.54) and (3.57), we get in $\mathcal{U}$

\begin{equation}
(3.58) \quad 4c + \beta \gamma = f \beta.
\end{equation}

From (3.17) we find, by the help of (3.57) and (3.58), $\beta(ie - f \gamma) = 0$ in $\mathcal{U}$. If $ie - f \gamma \neq 0$ at a point $Q$ in $\mathcal{U}$, then $\beta = 0$ at $Q$. This is impossible from (3.58). Hence we have in $\mathcal{U}$

\begin{equation}
(3.59) \quad ei = f \gamma.
\end{equation}

Substituting (3.57), (3.58) and (3.59) into (3.43), we have $c = 0$. This contradicts to our hypothesis $c \neq 0$. Thus we have a contradiction in the case of $c \neq 0$ in $\mathcal{U}$.

Summing up, if we assume that $\delta$ is always zero in $\mathcal{U}$, then we can deduce a contradiction.

Case B. Assume that $\delta$ is not zero at some point $Q$ of $\mathcal{U}$.

Then, $\delta$ is not zero in some open neighborhood $\mathcal{W}' \subset \mathcal{U}$ of $Q$. In this case, we can choose another local frame field $W$, $X'$, $\phi X'$ in the open neighborhood $\mathcal{W}'$ by

\begin{equation*}
X' = \cos \theta X + \sin \theta \phi X, \\
\phi X' = -\sin \theta X + \cos \theta \phi X,
\end{equation*}

where $\theta(0 < \theta < \frac{\pi}{2})$ is determined by $\cot 2\theta = \frac{\beta - \gamma}{2\delta}$. Then $\theta$ is a differentiable function in the open neighborhood and we have

\begin{align*}
AW &= aW + \left( b \cos \theta + e \sin \theta \right) X' + \left( -b \sin \theta + e \cos \theta \right) \phi X', \\
AX' &= \left( b \cos \theta + e \sin \theta \right) W + \left( \beta \cos^2 \theta + \gamma \sin^2 \theta + 2\delta \sin \theta \cos \theta \right) X', \\
A\phi X' &= \left( -b \sin \theta + e \cos \theta \right) W + \left( \beta \sin^2 \theta + \gamma \cos^2 \theta - 2\delta \sin \theta \cos \theta \right) \phi X'.
\end{align*}
Therefore we have the following form of equations in $\mathcal{W}'$ instead of (3.2)

\[ AW = \alpha W + b' X' + e' \phi X', \]

(3.60)

\[ AX' = b'W + \beta' X', \]

\[ A\phi X' = e'W + \gamma' \phi X'. \]

Since $W$ is not principal in $\mathcal{U}$, it is also not principal in $\mathcal{W}'$. Hence $b' \neq 0$ in an open neighborhood $\mathcal{W}'_1 \subset \mathcal{U}$ of $Q$ or $e' \neq 0$ in an open neighborhood $\mathcal{W}'_2 \subset \mathcal{U}$ of $Q$. Thus, $b' \neq 0$ in an open neighborhood $\mathcal{W} \subset \mathcal{U}$ of $Q$ or $e' \neq 0$ in an open neighborhood $\mathcal{W} \subset \mathcal{U}$ of $Q$. Since the function $\delta'$ corresponding to $\delta$ is always zero in $\mathcal{W}$, the situation is same to the Case A and we also have a contradiction.

Thus we have the following:

**Theorem 3.1.** There does not exist a Ricci-parallel real hypersurface in $P_2 \mathbb{C}$ or $H_2 \mathbb{C}$.

**References**


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