SETS OF WEAK EXPONENTS OF INDECOMPOSABILITY FOR IRREDUCIBLE BOOLEAN MATRICES

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ABSTRACT. Let $IB_n$ be the set of all irreducible matrices in $B_n$ and let $SIB_n$ be the set of all symmetric matrices in $IB_n$. Finding an upper bound for the set of indices of matrices in $IB_n$ and $SIB_n$ and determining gaps in the set of indices of matrices in $IB_n$ and $SIB_n$ has been studied by many researchers. In this paper, we establish a best upper bound for the set of weak exponents of indecomposability of matrices in $SIB_n$ and $IB_n$, and show that there does not exist a gap in the set of weak exponents of indecomposability for any of class $SIB_n$ and class $IB_n$.

1. Introduction

Let $B_n$ be the set of all $n \times n$ Boolean matrices; that is, all $(0,1)$-matrices with the usual arithmetic except that $1 + 1 = 1$. Let $r$ be an integer with $-n < r < n$. A matrix $A \in B_n$ is $r$-indecomposable if it contains no $k \times l$ zero submatrix with $1 \leq k, l \leq n$ and $k + l = n - r + 1$. In particular, $A$ is $(1 - n)$-indecomposable if and only if $A \neq 0$, while $A$ is $(n - 1)$-indecomposable if and only if $A = J_n$, the all-1’s matrix. A 1-indecomposable matrix is also said to be fully indecomposable, and a 0-indecomposable matrix is also called a Hall matrix.

By the definition of $r$-indecomposability, a matrix $A \in B_n$ is $r$-indecomposable if and only if, for each $k$ such that $\max\{1, 1 - r\} \leq k \leq \min\{n, n - r\}$, every $k \times n$ submatrix of $A$ has at least $k + r$ columns with nonzero entries.

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A matrix $A \in B_n$ is reducible if there is a permutation matrix $P$ such that

$$PAP^{-1} = \begin{pmatrix} A_1 & 0 \\ A_{21} & A_2 \end{pmatrix},$$

where $A_1$ and $A_2$ are nonvacuous square matrices; otherwise $A$ is irreducible. Note that for any $A \in IB_n$ with $n > 1$

$$A + A^2 + \cdots + A^n = J_n$$

and $J_n$ is $r$-indecomposable for any $r$ with $-n < r < n$. Hence for any $A \in IB_n$ and any integer $r$ with $-n < r < n$, there exists a minimum positive integer $p$ such that $A + A^2 + \cdots + A^p$ is $r$-indecomposable. Such an integer $p$ is called the weak exponent of $r$-indecomposability of $A$, and is denoted by $w_r(A)$.

Brualdi and Liu[3] used $f_w(A)$, $h_w(A)$ for $w_1(A)$ and $w_0(A)$ and called them the weak fully indecomposable exponent and weak Hall exponent of $A$ respectively. They suggested that further study on these weak exponents be done.

Liu[9] proved that $f_w(A) \leq \lceil \frac{n}{2} \rceil + 1$ and $h_w(A) \leq \lceil \frac{n}{2} \rceil$ for any $A \in IB_n$, where $[x]$ and $\lceil x \rceil$ denote the greatest integer $\leq x$ and the smallest integer $\geq x$ respectively.

In [1], it is proven that $w_r(A) \leq \lfloor (n + r + 1)/2 \rfloor$, for any matrix $A \in IB_n$ and integer $r$ with $-n < r < n$, and this upper bound is best possible.

Let $SIB_n$ be the set of all symmetric matrices in $IB_n$. It can easily be seen that for any $A \in SIB_n$ with $n > 2$, $A + A^2 + \cdots + A^{n-1} = J_n$, which is equivalent to $w_{n-1}(A) \leq n - 1$ (See [5]). Let $A \in B_n$. The sequence of powers $A^0 = I$, $A^1$, $A^2$, $\ldots$ forms a finite subsemigroup of $B_n$. Thus there is a least non-negative integer $k = k(A)$ and a least positive integer $p = p(A)$ such that $A^k = A^{k+p}$. The integers $k = k(A)$ and $p = p(A)$ are called the index of convergence or in short the index of $A$ and the period of $A$, respectively. A matrix in $IB_n$ with period 1 is called a primitive matrix. It is well known that $k(A) \leq n^2 - 2n + 2$ for any primitive matrix and the equality holds for the Wielandt matrix (see [4, p.82]).

The problem of finding an upper bound for the set of indices of a certain class of matrices in $IB_n$ and $SIB_n$ and determining gaps in the set of indices of a certain class of matrices in $IB_n$ and $SIB_n$ has been studied by many researchers. (See [6, 11, 13]. They especially studied the class of primitive matrices.) In this paper, we establish a best upper bound for the set of weak exponents of $r$-indecomposability of matrices
in $SIB_n$ and $IB_n$, and show that there is no gap in the set of weak exponents of $r$-indecomposability for any of class $SIB_n$ and class $IB_n$.

2. Results

For a matrix $A = (a_{ij}) \in B_n$, the directed graph of $A$, $D(A)$, is the graph with vertex set $V(D(A)) = \{1, 2, \cdots, n\}$ and arc set $E(D(A)) = \{(i, j) : a_{ij} \neq 0\}$. It is well known that the $(i, j)$ entry of $A^k$ is nonzero if and only if there is a walk of length $k$ from vertex $i$ to vertex $j$ in $D(A)$. If $A \in B_n$ is symmetric, then $D(A)$ corresponds naturally a graph $D_G(A)$ by replacing arcs $(u, v)$ and $(v, u)$ by an edge $uv$.

The following theorem provides a best possible upper bound for the set of weak exponents of $r$-indecomposability of matrices in $SIB_n$.

**Theorem 1.** For any matrix $A \in SIB_n$ with $n > 2$, and any integer $r$ with $-n < r < n$, we have

$$w_{1-n}(A) = w_{2-n}(A) = 1;$$

$$w_r(A) \leq \begin{cases} r & \text{if } 2 \leq r \leq n-1, \\ 2 & \text{if } 3-n \leq r \leq 1, \end{cases}$$

and this bound is best possible.

**Proof.** We consider the following three cases: $r = 1 - n$ or $2 - n$; $2 \leq r \leq n - 1$; $3 - n \leq r \leq 1$.

Suppose that $r = 1 - n$ or $2 - n$. For any $A \in SIB_n$, $A$ has neither zero rows nor zero columns, implying that $A$ is $r$-indecomposable. So $w_r(A) = 1$.

Suppose that $2 \leq r \leq n - 1$. Assume that $A + A^2 + \cdots + A^r$ is not $r$-indecomposable. Then it contains a $k \times l$ zero submatrix with $1 \leq k, l \leq n$ and $k + l = n - r + 1$. Let $D = D(A)$. Then there are subsets $V_1, V_2 \subseteq V(D)$ with $|V_1| = k, |V_2| = l$ such that for any integer $m$ with $1 \leq m \leq r$, there is no walk of length $m$ from any vertex in $V_1$ to any vertex in $V_2$. Since $A$ is symmetric, $V_1 \cap V_2 = \emptyset$. On the other hand, by the strong connectivity of $D$, there is a vertex $u \in V_1$ and a vertex $v \in V_2$ such that the distance from $u$ to $v$ is at most $n - |V_1| - |V_2| + 1 = n - (n - r + 1) + 1 = r$, which is a contradiction. So $A + A^2 + \cdots + A^r$ is $r$-indecomposable and $w_r(A) \leq r$.

In order to show the sharpness of the bound, take $A_0 \in SIB_n$ where $D_G(A_0)$ is the path on $n$ vertices $1, 2, \ldots, n$ with edges $i(i+1)$, $i = 1, 2, \ldots, n-1$. It is easy to see that the $1 \times (n-r+1)$ submatrix indexed
by the first row and the last \( n - r \) columns in \( A_0 + A_0^2 + \cdots + A_0^{r-1} \) is zero. This implies that \( w_r(A_0) \geq r \). Hence \( w_r(A_0) = r \).

Finally suppose that \( 3 - n \leq r \leq 1 \). Note that an \( r \)-indecomposable matrix is also \((r - 1)\)-indecomposable. In this case, \( w_r(A) \leq w_1(A) \leq w_2(A) \leq 2 \).

To show the bound is best possible, take \( A_0 \in SIB_n \), where \( D_G(A_0) \) is the star \( K_{1,n-1} \). Clearly \( w_r(A) = 2 \).

The following theorem completely determines the set of weak exponents of \( r \)-decomposability of matrices in \( SIB_n \).

**Theorem 2.** Let \( w_r(SIB_n) = \{w_r(A) : A \in SIB_n\} \) with \( n > 2 \). Then

\[
w_r(SIB_n) = \begin{cases} 
\{1\} & \text{if } r = 1 - n, 2 - n, \\
\{1, 2\} & \text{if } 3 - n \leq r \leq 1, \\
\{1, 2, \ldots, r\} & \text{if } 2 \leq r \leq n - 1.
\end{cases}
\]

**Proof.** Note that \( J_n \in SIB_n \), \( w_r(J_n) = 1 \) for all \( 1 - n \leq r \leq n - 1 \). The case \( 1 - n \leq r \leq 2 \) follows from Theorem 1. Note also that \( w_r(A_0) = 2 \) for all \( 3 \leq r \leq n - 1 \), where \( D_G(A_0) \) is the star \( K_{1,n-1} \). Suppose \( 3 \leq r \leq n - 1 \). By Theorem 1 we only need to show that \( \{3, \ldots, r-1\} \subseteq w_r(SIB_n) \) for \( 3 \leq r \leq n - 1 \).

For any integer \( 3 \leq k \leq r-1 \), take \( A_1 \in SIB_n \), where \( D_G(A_1) = G \) is a graph on vertices \( 1, 2, \ldots, n \) with edges \( i(n-k+1), i = 1, 2, \ldots, n-k \) and \( i(i+1), i = n-k+1, \ldots, n \). It is easy to see that \( A_1 + A_1^2 + \cdots + A_1^{k-1} \) contains an \((n-k) \times 1\) zero submatrix, so \( A_1 \) is not \( k \)-indecomposable and hence not \( r \)-indecomposable. But \( A_1 + A_1^2 + \cdots + A_1^{k-1} = J_n \). We have \( w_r(A_1) = k \), and hence \( \{3, \ldots, r-1\} \subseteq w_r(SIB_n) \) for \( 3 \leq r \leq n - 1 \).

Let \( A \in B_n \) and let \( X \subseteq V(D(A)) \). By \( R_t(A, X) \), we denote the set of all vertices reachable from a vertex in \( X \) via a walk of length \( t \). Clearly, \( R_1(A^t, X) = R_t(A, X) \). Then \( A \in B_n \) is \( r \)-indecomposable if and only if, for each \( X \subseteq V(D(A)) \) with \( \max\{1, 1 - r\} \leq |X| \leq \min\{n, n - r\} \), \( |R_1(A, X)| \geq |X| + r \).

The following theorem completely determines the set of weak exponents of \( r \)-decomposability of matrices in \( IB_n \). We need the following Lemma to prove it.

**Lemma 3** ([1, Lemma 1], [9]). Suppose that \( A \in IB_n \), \( X \subseteq V(D(A)) \), and \( 1 \leq t \leq n \). If \( R_1(\sum_{i=1}^t A^i, X) \neq V(D(A)) \), then

\[
|R_1(\sum_{i=1}^t A^i, X)| \geq |R_1(A, X)| + t - 1.
\]
THEOREM 4. Let \( w_r(IB_n) = \{w_r(A) : A \in IB_n\} \) with \(-n < r < n\), \(n > 1\). Then

\[
w_r(IB_n) = \left\{1, 2, \ldots, \left\lfloor \frac{n + r + 1}{2} \right\rfloor \right\}.
\]

Proof. Note that \( [1] w_r(A) \leq [(n + r + 1)/2] \) for any \( A \in IB_n \). The case \( r = 1 - n, 2 - n \) is trivial. Suppose in the following \( 3 - n \leq r \leq n - 1 \). We need only to show that

\[
\{1, 2, \ldots, [(n + r + 1)/2]\} \subseteq w_r(IB_n).
\]

For integer \( a \) with \( \max\{1 - r, 1\} \leq a \leq [(n - r + 1)/2] \), take \( A_0 \in IB_n \) with \( D(A) = D \), where \( V(D) = \{1, 2, \ldots, n\} \) and \( E(D) = \{(i, a + 1) : 1 \leq i \leq a\} \cup \{(i, i + 1) : a + 1 \leq i \leq n - 1\} \cup \{(n, i) : 1 \leq i \leq a\} \). It can be easily seen that all columns except columns \( a + 1, \ldots, 2a + r - 1 \) are zero in rows \( 1, 2, \ldots, a \) of \( A_0 + A_0^2 + \cdots + A_0^{a+r-1} \); hence \( A_0 + A_0^2 + \cdots + A_0^{a+r-1} \) contains a \( a \times (n - a - r + 1) \) zero submatrix with \( a + (n - a - r + 1) = n - r + 1 \), which implies that \( w_r(A_0) \geq a + r \). It can be checked that for each \( X \subseteq V(D) \) with \( \max\{1, 1 - r\} \leq |X| \leq \min\{n, n - r\} \),

\[
|R_1(A_0, X)| \geq |X| - a + 1,
\]

and hence, by Lemma 3, \( |R_1(A_0 + A_0^2 + \cdots + A_0^{a+r}, X)| \geq |R_1(A_0, X)| + a + r - 1 \geq |X| + r \). This implies that \( A_0 + A_0^2 + \cdots + A_0^{a+r} \) is \( r \)-indecomposable. We have \( w_r(A_0) = a + r \).

Suppose that \( 3 - n \leq r \leq -1 \). We take \( a = 1 - r, 2 - r, \ldots, [(n - r + 1)/2] \) to obtain \( \{1, 2, \ldots, [(n + r + 1)/2]\} \subseteq w_r(IB_n) \).

If \( 1 \leq r \leq n - 1 \), then we first take \( a = 1, 2, [(n - r + 1)/2] \) to obtain \( \{r + 1, r + 2, \ldots, [(n + r + 1)/2]\} \subseteq w_r(IB_n) \). Then by Theorem 2, we have \( \{1, 2, \ldots, r\} \subseteq w_r(IB_n) \).

In either case, \( \{1, 2, \ldots, [(n + r + 1)/2]\} \subseteq w_r(IB_n) \). It completes the proof. \( \square \)

3. Closing remark

Let \( A \in B_n \). If there exists a positive integer \( k \) such that \( A^k \) is \( r \)-indecomposable, then the smallest positive integer \( k \) is called the exponent of \( r \)-indecomposability of \( A \). If there exists a positive integer \( k \) such that \( A^i \) is \( r \)-indecomposable for all \( i \geq k \), then the smallest positive integer \( k \) is called the strict exponent of \( r \)-indecomposability of \( A \). These (strict) exponents of \( r \)-indecomposability of primitive matrices have been investigated in [7, 12]. The cases when \( r = 1 \) (fully indecomposable exponent) and \( r = 0 \) (Hall exponent) have already been studied.
extensively (see [2, 3, 8, 10]). In this paper, we studied weak exponent of $r$-indecomposability of irreducible matrices some of whose special cases are the weak fully indecomposable exponent and weak Hall exponent initiated by Brualdi and Liu [3]. Theorems 2 and 4 tell us that there is no gap in the set of weak exponents of $r$-indecomposability for any of class $SIB_n$ and class $IB_n$.

References


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