A STRUCTURE THEOREM
FOR COMPLETE INTERSECTIONS

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ABSTRACT. Buchsbaum and Eisenbud proved a structure theorem for Gorenstein ideals of grade 3. In this paper we derive a class of the perfect ideals from a class of the complete matrices. From this we give a structure theorem for complete intersections of grade $g > 3$.

1. Introduction

Let $R$ be a Noetherian local ring and $I$ a perfect ideal of grade $g$ in $R$. Many people have been studying the algebra structure on the minimal free resolution of $R/I$, in particular, Gorenstein ideals, the ideals of type 1. In 1968, Burch [3] characterized perfect ideals of grade 2 by showing a structure theorem due to Hilbert in a special case: every perfect ideal of grade 2 generated by $n$ elements is the ideal of $(n - 1)$st order minors of an $(n - 1) \times n$ matrix. In 1977, Buchsbaum and Eisenbud [2] gave a structure theorem for Gorenstein ideals of grade 3 which says that every Gorenstein ideal of grade 3 in $R$ is generated by the maximal order Pfaffians of an alternating matrix. However a structure theorem for Gorenstein ideals of grade 4 is more complicated than that of grade 3 and not completely known. In 1987, Brown [1] described a structure theorem for a certain class of perfect ideals $I$ which have grade 3, type 2 and $\lambda(I) = \dim_k \Lambda^2_1 > 0$, where $\lambda(I)$ is a numerical invariant defined in [5]. In 1989, Sanchez [7] gave a structure theorem for type 3, grade 3 perfect ideals which have $\lambda(I) = \dim_k \Lambda^2_1 = 2$ or greater. In this paper we will describe a structure theorem for complete intersections of grade $g > 3$, which says that every complete intersection of grade $g > 3$ in $R$ is generated by the elements $x_i$’s, where $x_i^{g-1}$ is the determinant of the $(g - 1) \times (g - 1)$ diagonal matrix drawn from a complete matrix of grade $g$ for each $i$ ($1 \leq i \leq g$).

In Section 2 we review some of the properties of alternating matrices, linkage theory, and a structure theorem for Gorenstein ideals of grade 3.

In Section 3 we give the concept of a complete matrix of grade 4 and provide a structure theorem for complete intersections of grade 4.
In Section 4 we introduce a complete matrix $f$ of grade $g > 3$, and define the ideal $K_{g-1}(f)$ associated with $f$. Then we prove a structure theorem for complete intersections of grade $g > 3$. The structure theorem [4] for complete intersections of grade 4 is just a special case of our main Theorem 4.10. Throughout this paper, we assume that all rings are a Noetherian local ring with maximal ideal $m$ unless otherwise stated.

2. Gorenstein ideals of grade 3

The grade of a proper ideal $I$ in $R$ is the length of the maximal $R$-sequence contained in $I$. We say that an ideal $I$ of grade $g$ is perfect if grade $I = \text{projdim}_R(R/I) = g$. If $I$ is a perfect ideal of grade $g$, then the type of $I$ is defined to be the dimension of the $R/m$-vector space $\text{Ext}_R^g(R/m, R/I)$. A perfect ideal $I$ of grade $g$ is Gorenstein if type $I = 1$, equivalently, if $F$ is the minimal free resolution of $R/I$,

$$0 \rightarrow F_g \xrightarrow{\varphi_g} F_{g-1} \xrightarrow{\varphi_{g-1}} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 (= R),$$

then the rank of $F_g$ is 1. A perfect ideal $I$ of grade $g$ is a complete intersection if it is generated by $g$ elements, and is an almost complete intersection if it is minimally generated by $g + 1$ elements.

Let $R$ be a commutative ring, and $F$ a finite free $R$-module. An $R$-module homomorphism $\varphi : F \rightarrow F^*$ is said to be alternating if with respect to some (and therefore any) basis of $F$ and the corresponding dual basis of $F^*$, the matrix $\varphi$ is alternating, i.e., skew-symmetric and all its diagonal entries are 0. Now suppose that $\varphi$ is alternating, choose a basis of $F$ and the corresponding dual basis of this, and identify $\varphi$ with the corresponding matrix $(\varphi_{ij})$. If rank $F$ is odd, then $\det \varphi = 0$, and if rank $F$ is even, then there exists an element $\text{Pf}(\varphi) \in R$, called the Pfaffian of $\varphi$, which is a polynomial function of the entries of $\varphi$, such that $\det \varphi = \text{Pf}(\varphi)^2$. We set $\text{Pf}(\varphi) = 0$ if rank $F$ is odd.

Pfaffians can be developed along a row just like the determinants. Denote by $\text{Pf}_r(\varphi)$ the ideal generated by the $r$th order Pfaffians of $\varphi$. With these concepts Buchsbaum and Eisenbud gave a complete structure for Gorenstein ideals of grade 3:

**Theorem 2.1** ([2]). Let $R$ be a Noetherian local ring with maximal ideal $m$.

1. Let $F$ be a free $R$-module with rank $F = n$, where $n \geq 3$ is an odd integer. Let $\varphi : F^* \rightarrow F$ be an alternating map whose image is contained in $mF$. Suppose that $\text{Pf}_{n-1}(\varphi)$ has grade 3. Then $\text{Pf}_{n-1}(\varphi)$ is a Gorenstein ideal minimally generated by $n$ elements.

2. Every Gorenstein ideal of grade 3 arises as in (1).

Now we review some of the notions in the linkage theory formulated by Peskine and Szpiro in [6].

**Definition 2.2.** Let $I$ and $J$ be two ideals in a Gorenstein ring $R$ (not necessarily local).
(1) If there exists an $R$-regular sequence $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_g$ in $I \cap J$ such that $J = (\alpha) : I$ and $I = (\alpha) : J$, then $I$ and $J$ are said to be linked (with respect to $\alpha$).

(2) If $I$ and $J$ are linked and if $\text{Ass}(R/I) \cap \text{Ass}(R/J) = \emptyset$, equivalently, if $I$ and $J$ are linked (with respect to $\alpha$) and if $I \cap J = (\alpha)$, then $I$ and $J$ are said to be geometrically linked.

Let $R$ be a Gorenstein local ring of Krull dimension $g$ with maximal ideal $m$. If $I$ and $J$ are perfect ideals of grade $g$, then they are not geometrically linked because $(R/I)$ and $(R/J)$ are both zero-dimensional artinian local rings. Peskine and Szpiro gave a method of constructing a Gorenstein ideal of grade $g + 1$ from two perfect ideals of grade $g$:

**Theorem 2.3** ([6]). Let $R$ be a Gorenstein local ring with maximal ideal $m$. Let $I$ and $J$ be geometrically linked Cohen-Macaulay ideals of grade $g$ by a regular sequence $x = x_1, x_2, \ldots, x_g$ and let $K = I + J$. Then $K$ is a Gorenstein ideal of grade $g + 1$.

Let $F$ be a free $R$-module with a basis $\{e_1, e_2, \ldots, e_n\}$ and let $I$ be an ideal generated by a regular sequence $x = x_1, x_2, \ldots, x_n$. Let $K(x)$ be the Koszul complex defined by $x = x_1, x_2, \ldots, x_n$. Then

$$K(x) : 0 \longrightarrow \wedge^n F \xrightarrow{d_n} \wedge^{n-1} F \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} \wedge^1 F \xrightarrow{d_1} \wedge^0 F$$

is the minimal free resolution of $R/I$, where $d_1(e_i) = x_i$ for each $i$ with $1 \leq i \leq n$, and for each $p$ with $1 \leq p \leq n$, $d_p : \wedge^p F \rightarrow \wedge^{p-1} F$ is given by

$$(2.1) \quad d_p(e_i_1 \wedge e_i_2 \wedge \cdots \wedge e_i_p) = \sum_{j=1}^p (-1)^{j-1} d_1(e_i_j)e_i_1 \wedge e_i_2 \wedge \cdots \wedge \hat{e}_{i_j} \wedge \cdots \wedge e_i_p.$$ 

For example, if $x = x_1, x_2, x_3, x_4, x_5$ is a regular sequence on $R$, then $d_2$ has the form

$$(2.2) \quad d_2 = \begin{bmatrix}
-x_2 & -x_3 & -x_4 & -x_5 & 0 & 0 & 0 & 0 & 0 \\
x_1 & 0 & 0 & 0 & -x_3 & -x_4 & -x_5 & 0 & 0 \\
0 & x_1 & 0 & 0 & x_2 & 0 & 0 & -x_4 & -x_5 \\
0 & 0 & x_1 & 0 & 0 & x_2 & 0 & x_3 & -x_5 \\
0 & 0 & 0 & x_1 & 0 & 0 & x_2 & 0 & x_3 \\
0 & 0 & 0 & 0 & x_1 & 0 & 0 & x_2 & x_4
\end{bmatrix}.$$ 

The exterior algebra $\wedge F$ is a graded Hopf algebra such that $x \wedge y = (-1)^{pq} y \wedge x$ for $x \in \wedge^p F$ and $y \in \wedge^q F$ and $x \wedge x = 0$ for any homogeneous element $x$ of odd degree. It is well-known that the algebra structure on the Koszul complex which gives the minimal free resolution of a complete intersection is an exterior algebra.
3. Complete intersections of grade 4

In this section we start with a skew-symmetrizable matrix, and a complete matrix of grade 4 which play important roles in describing the complete intersections of grade 4.

**Definition 3.1.** Let $R$ be a commutative ring with identity. An $n \times n$ matrix $X = (x_{ij})$ over $R$ is said to be generalized alternating or skew-symmetrizable if there exist nonzero $n \times n$ diagonal matrices $D' = \text{diag}(u_1, u_2, \ldots, u_n)$ and $D = \text{diag}(w_1, w_2, \ldots, w_n)$ with entries in $R$ such that $D'XD$ is alternating. We denote by $\text{GA}_n(R)$ the set of all skew-symmetrizable $n \times n$ matrices over $R$. If there is no ambiguity about the ring $R$, then $\text{GA}_n(R)$ is denoted by $\text{GA}_n$.

Notice that every alternating matrix is skew-symmetrizable. For an $n \times n$ skew-symmetrizable matrix $X$, we denote $\mathcal{A}(X)$ to be an alternating matrix $D'XD$ for some diagonal matrices $D'$ and $D$. To define a complete intersection of grade 4, we need to describe the submatrices of the given matrix in detail. A $p \times q$ submatrix of an $m \times n$ matrix $f$ is a matrix obtained from $f$ by taking the $pq$ entries at the intersections of the $i_1$th, $i_2$th, $\ldots$, $i_p$th rows and the $j_1$th, $j_2$th, $\ldots$, $j_q$th columns of $f$, where $1 \leq i_1 < i_2 < \cdots < i_p \leq m$ and $1 \leq j_1 < j_2 < \cdots < j_q \leq n$. The corresponding $p \times q$ submatrix of $f$ is denoted by

$$f(i_1, i_2, \ldots, i_p | j_1, j_2, \ldots, j_q).$$

Notice that the $p \times q$ matrix $f(i_1, i_2, \ldots, i_p | j_1, j_2, \ldots, j_q)$ consisting of the $pq$ entries at the intersection of these rows and columns of $f$ could not be a submatrix of $f$ unless $1 \leq i_1 < i_2 < \cdots < i_p \leq m$ and $1 \leq j_1 < j_2 < \cdots < j_q \leq n$.

Next we get into the skew-symmetrizable matrices and the special properties of the second differential map $d_2$ of the Koszul complex $\mathbb{K}(\mathbf{x})$.

**Proposition 3.2.** With the notation as above, the second differential map $d_2$ of the Koszul complex satisfies the following properties:

1. There are four disjoint pairs $(S, T)$ of two $4 \times 3$ submatrices of $d_2$.
2. By removing a row and interchanging columns, each pair $(S, T)$ can be reduced to a pair $(\bar{S}, \bar{T})$ of $3 \times 3$ matrices such that $\bar{S}$ is a diagonal matrix whose determinant is the nonzero 3rd power element $x^3$ for some $x \in R$, and $\bar{T}$ is a skew-symmetrizable matrix with grade $\text{Pf}_2(\mathcal{A}(\bar{T})) = 3$. 
Let \( P_f \) denote the determinant of a skew-symmetrizable matrix \( \mathcal{A} \) with grade \( d \). Then we consider two submatrices of \( \mathcal{A} \) with grade \( d \). Removing the first row and interchanging columns 1 and 3 of \( T_1 \), we have the \( 3 \times 3 \) matrix \( T_1 \). Then \( T_1 \) is skew-symmetrizable, since it becomes an alternating matrix by multiplying the second column of it by \(-1\). Since \( x_2, x_3, x_4 \) is a regular sequence on \( R \), \( P_{\mathcal{A}(T_1)}(x_2, x_3, x_4) \) has grade 3. Similarly, we can take the disjoint submatrices of \( d_2 \):

\[
\begin{align*}
S_2 &= d_2(1, 2, 3, 4 \mid 1, 4, 5) \text{ and } T_2 = d_2(1, 2, 3, 4 \mid 2, 3, 6), \\
S_3 &= d_2(1, 2, 3, 4 \mid 2, 4, 6) \text{ and } T_3 = d_2(1, 2, 3, 4 \mid 1, 3, 5), \\
S_4 &= d_2(1, 2, 3, 4 \mid 3, 5, 6) \text{ and } T_4 = d_2(1, 2, 3, 4 \mid 1, 2, 3).
\end{align*}
\]

The similar argument gives us the \( 3 \times 3 \) diagonal matrix \( \bar{S} \) whose determinant is equal to \( x_i^3 \) or \((-x_i)^3\), and the \( 3 \times 3 \) skew-symmetrizable matrix \( \bar{T} \) with grade \( P_{\mathcal{A}(\bar{T})} = 3 \) for \( i = 2, 3, 4 \).

**Definition 3.3.** Let \( R \) be a commutative ring with identity. A \( 4 \times 6 \) matrix \( f \) over \( R \) is said to be a complete matrix of grade 4 if

1. \( f \) has four distinct pairs \((S, T)\) of disjoint \( 4 \times 3 \) submatrices;
2. By removing a row and interchanging columns, each pair \((S, T)\) is reduced to a pair \((\bar{S}, \bar{T})\) of \( 3 \times 3 \) matrices such that \( \bar{S} \) is a diagonal matrix whose determinant is a nonzero 3rd power element \( x^3 \) for some \( x \in R \), and \( \bar{T} \) is a skew-symmetrizable matrix with grade \( P_{\mathcal{A}(\bar{T})} = 3 \).

The following example illustrates Definition 3.3.

**Example 3.4.** Let \( x, y, z, \) and \( w \) be a regular sequence on a commutative ring \( R \). Let \( f \) be a \( 4 \times 6 \) matrix given by

\[
f = \begin{bmatrix}
0 & 0 & -y & -w & -z & 0 \\
0 & -z & x & 0 & 0 & -w \\
-w & y & 0 & 0 & x & 0 \\
0 & 0 & x & 0 & y & 0
\end{bmatrix}.
\]

Then \( f \) is a complete matrix of grade 4. To see this, we find four distinct pairs of disjoint \( 4 \times 3 \) submatrices \( S_i \) and \( T_i \) of \( f \) satisfying the properties in Proposition 3.2. First we consider two submatrices of \( f \):

\[
S_1 = \begin{bmatrix}
-y & -w & -z \\
x & 0 & 0 \\
0 & 0 & x \\
0 & x & 0
\end{bmatrix} \quad \text{and} \quad T_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & -z & -w \\
-w & y & 0 \\
z & 0 & y
\end{bmatrix},
\]

that is, \( S_1 = f(1, 2, 3, 4 \mid 3, 4, 5) \) and \( T_1 = f(1, 2, 3, 4 \mid 1, 2, 6) \). So \( S_1 \) and \( T_1 \) are disjoint. By removing the first row and interchanging the second and the third columns of \( S_1 \) and \( T_1 \), we can get the \( 3 \times 3 \) matrices \( S_1 = S_1(2, 3, 4 \mid 1, 3, 2) \) and \( T_1 = T_1(2, 3, 4 \mid 1, 3, 2) \). Then \( S_1 \) is a diagonal matrix whose determinant is a nonzero element \( x^3 \) and \( T_1 \) is skew-symmetrizable since \( T_1 \text{diag}(1, -1, 1) \).
Let $S = f(1, 2, 3, 4 | 2, 3, 6)$ and $T = f(1, 2, 3, 4 | 1, 4, 5),
S = f(1, 2, 3, 4 | 1, 2, 5)$ and $T = f(1, 2, 3, 4 | 3, 4, 6),
S = f(1, 2, 3, 4 | 1, 4, 6)$ and $T = f(1, 2, 3, 4 | 2, 3, 5).

Clearly, $4 \times 3$ submatrices $S_i$ and $T_i$ of $f$ are disjoint for $i = 2, 3, 4$. The similar argument gives us the following $3 \times 3$ matrices:

$S_2 = S_2(1, 3, 4 | 2, 1, 3)$ and $T_2 = T_2(1, 3, 4 | 1, 2, 3),
S_3 = S_3(1, 2, 4 | 3, 2, 1)$ and $T_3 = T_3(1, 2, 4 | 3, 2, 1),
S_4 = S_4(1, 2, 3 | 2, 3, 1)$ and $T_4 = T_4(1, 2, 3 | 1, 3, 2).

And $\det S_2 = (-y)^3$, $\det S_3 = z^3$ and $\det S_4 = (-w)^3$ are nonzero 3rd power elements and

$\text{Pf}_2(A(T_1)) = (y, z, w)$, $\text{Pf}_2(A(T_2)) = (x, z, w),
\text{Pf}_2(A(T_3)) = (x, y, w), \text{Pf}_2(A(T_4)) = (x, y, z).

Since $x, y, z, w$ is a regular sequence on $R$, these four ideals have all grade 3. Hence the properties in Proposition 3.2 are satisfied.

We notice that if $f$ is a complete matrix of grade 4, then the matrix obtained from $f$ by interchanging rows of $f$ also becomes a complete matrix of grade 4.

**Theorem 3.5** ([4]). Let $f = \{f_{ij}\}$ be a $4 \times 6$ complete matrix of grade 4.

1. Every column of $f$ has exactly two nonzero entries.
2. The number of nonzero rows in each $4 \times 2$ submatrix of $f$ is greater than 2.
3. Each pair $(S, T)$ of $3 \times 3$ matrices given in Definition 3.3 is uniquely determined.

Now we will define an ideal $K_3(f)$ generated by the radical roots of the determinants of the $3 \times 3$ diagonal matrices $S$ derived from a given complete matrix $f$ of grade 4 in Theorem 3.5.

**Definition 3.6.** Let $f$ be a $4 \times 6$ complete matrix of grade 4. Let $S_i$ be a unique $3 \times 3$ diagonal matrix reduced from the disjoint pair $(S_i, T_i)$ of $f$ such that $\det S_i = x_i^3$ is nonzero for $i = 1, 2, 3, 4$. We define $K_3(f)$ to be the ideal generated by the $x_i$'s, that is,

$$K_3(f) = (x_1, x_2, x_3, x_4).$$

Next let us show that the ideal $K_3(f)$ defines a complete intersection of grade 4. Let $f$ be a complete matrix of grade 4. By Theorem 3.5 we may...
Assume
\[ f = \begin{bmatrix}
  f_{11} & f_{12} & f_{13} & 0 & 0 & 0 \\
  f_{21} & 0 & 0 & f_{24} & f_{25} & 0 \\
  0 & f_{32} & 0 & f_{34} & 0 & f_{36} \\
  0 & 0 & f_{43} & 0 & f_{45} & f_{46}
\end{bmatrix}. \]

Then we have
\[ S_1 = f(2, 3, 4|1, 2, 3) \quad \text{and} \quad T_1 = f(2, 3, 4|6, 5, 4), \]
\[ S_2 = f(1, 3, 4|1, 4, 5) \quad \text{and} \quad T_2 = f(1, 3, 4|6, 3, 2), \]
\[ S_3 = f(1, 2, 4|2, 4, 6) \quad \text{and} \quad T_3 = f(1, 2, 4|5, 3, 1), \]
\[ S_4 = f(1, 2, 3|3, 5, 6) \quad \text{and} \quad T_4 = f(1, 2, 3|4, 2, 1), \]
i.e.,
\[ S_1 = \begin{bmatrix}
  f_{21} & 0 & 0 \\
  0 & f_{32} & 0 \\
  0 & 0 & f_{43}
\end{bmatrix} \quad \text{and} \quad T_1 = \begin{bmatrix}
  0 & f_{25} & f_{24} \\
  f_{36} & 0 & f_{34} \\
  f_{46} & f_{45} & 0
\end{bmatrix}, \]
\[ S_2 = \begin{bmatrix}
  f_{11} & 0 & 0 \\
  0 & f_{34} & 0 \\
  0 & 0 & f_{45}
\end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix}
  0 & f_{13} & f_{12} \\
  f_{36} & 0 & f_{32} \\
  f_{46} & f_{43} & 0
\end{bmatrix}, \]
\[ S_3 = \begin{bmatrix}
  f_{12} & 0 & 0 \\
  0 & f_{24} & 0 \\
  0 & 0 & f_{46}
\end{bmatrix} \quad \text{and} \quad T_3 = \begin{bmatrix}
  0 & f_{13} & f_{11} \\
  f_{25} & 0 & f_{21} \\
  f_{45} & f_{43} & 0
\end{bmatrix}, \]
\[ S_4 = \begin{bmatrix}
  f_{13} & 0 & 0 \\
  0 & f_{25} & 0 \\
  0 & 0 & f_{36}
\end{bmatrix} \quad \text{and} \quad T_4 = \begin{bmatrix}
  0 & f_{12} & f_{11} \\
  f_{24} & 0 & f_{21} \\
  f_{34} & f_{32} & 0
\end{bmatrix}. \]

Since \( T_i \text{diag}(u_{i_1}, u_{i_2}, u_{i_3}) \) is alternating where \( u_{i_k} \in \{ \pm 1 \} \), we have the following identities
\[ f_{24} = f_{46} \quad \text{or} \quad -f_{46}, \quad f_{25} = f_{36} \quad \text{or} \quad -f_{36}, \quad f_{34} = f_{45} \quad \text{or} \quad -f_{45}, \]
\[ f_{12} = f_{46} \quad \text{or} \quad -f_{46}, \quad f_{13} = f_{36} \quad \text{or} \quad -f_{36}, \quad f_{14} = f_{43} \quad \text{or} \quad -f_{43}, \]
\[ f_{11} = f_{45} \quad \text{or} \quad -f_{45}, \quad f_{13} = f_{25} \quad \text{or} \quad -f_{25}, \quad f_{12} = f_{43} \quad \text{or} \quad -f_{43}, \]
\[ f_{11} = f_{34} \quad \text{or} \quad -f_{34}, \quad f_{12} = f_{24} \quad \text{or} \quad -f_{24}, \quad f_{21} = f_{32} \quad \text{or} \quad -f_{32}. \]

Thus (3.2) and (3.3) give us
\[ \det S_1 = f_{21}f_{32}f_{43} = f_{21}^3 \quad \text{or} \quad -f_{21}^3, \quad \det S_2 = f_{11}f_{34}f_{45} = f_{11}^3 \quad \text{or} \quad -f_{11}^3, \]
\[ \det S_3 = f_{12}f_{24}f_{46} = f_{12}^3 \quad \text{or} \quad -f_{12}^3, \quad \det S_4 = f_{13}f_{25}f_{36} = f_{13}^3 \quad \text{or} \quad -f_{13}^3, \]
and
\[ \text{Pf}_2(A(T_1)) = (f_{11}, f_{12}, f_{13}), \quad \text{Pf}_2(A(T_2)) = (f_{21}, f_{13}, f_{12}), \]
\[ \text{Pf}_2(A(T_3)) = (f_{21}, f_{13}, f_{11}), \quad \text{Pf}_2(A(T_4)) = (f_{21}, f_{12}, f_{11}). \]
Hence
\[
Pf_2(A(T_i)) \subseteq K_3(f) = (f_{21}, f_{11}, f_{12}, f_{13}) \quad \text{for} \quad i = 1, 2, 3, 4.
\]
Thus we obtain the structure theorem for complete intersections of grade 4.

**Theorem 3.7 ([4]).** Let \( R \) be a Noetherian local ring with maximal ideal \( m \).

1. Let \( F \) and \( G \) be free \( R \)-modules with rank \( F = 6 \) and rank \( G = 4 \). Let \( f = (f_{ij}) : F \rightarrow G \) be a complete matrix of grade 4 such that \( \text{Im} \ f \subseteq mG \). With the notation as in Theorem 3.5, we assume that \( Pf_2(A(T_i)) + Pf_2(A(T_j)) \) has grade 4 for some \( i, j \) (\( i \neq j \)). Then the ideal \( K_3(f) \) is a complete intersection of grade 4.

2. Let \( I = (x_1, x_2, x_3, x_4) \) be a complete intersection of grade 4 and let

\[
\mathbb{F} : 0 \longrightarrow R \longrightarrow R^4 \longrightarrow R^6 \longrightarrow \cdots \longrightarrow R^4 \longrightarrow \phi_1 \longrightarrow R
\]
be the minimal free resolution of \( R/I \). Then \( \phi_2 \) and the transpose of \( \phi_3 \) satisfy the part (1).

### 4. Complete intersections of grade \( g > 4 \)

In this section we construct the ideal \( K_g(f) \) associated with a complete matrix \( f \) of grade \( g > 3 \) and provide a structure theorem for complete intersections of grade \( g > 3 \). We begin this section with easy lemmas.

**Lemma 4.1.** Let \( R \) be a Noetherian local ring with maximal ideal \( m \). For any positive integer \( g > 3 \), let \( x = x_1, x_2, \ldots, x_g \) and \( y_i = x_1, x_2, \ldots, \hat{x}_i, \ldots, x_g \) be regular sequences on \( R \), where \( \hat{x}_i \) indicates that \( x_i \) is to be omitted. Let \( K(x) \) and \( K(y_i) \) be the Koszul complexes of \( R/(x) \) and \( R/(y_i) \) for each \( i = 1, 2, 3, \ldots, g \).

Let
\[
K(x_i) : 0 \longrightarrow R \longrightarrow x_i \longrightarrow R
\]
be a complex of free \( R \)-modules and \( R \)-maps. Then

1. \( K(x) \cong K(x_i) \otimes K(y_i) \).

2. Let

\[
K(y_i) : 0 \longrightarrow F_{g-1} \longrightarrow F_{g-2} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow R,
\]
and

\[
K(x_i) \otimes K(y_i) : 0 \longrightarrow R \otimes F_{g-1} \longrightarrow R \otimes F_{g-2} \oplus R \otimes F_{g-1} \longrightarrow \cdots \longrightarrow R \otimes R
\]

\[
\longrightarrow R \otimes F_1 \longrightarrow R \otimes F_1 \longrightarrow R \otimes R.
\]
Then we have

$$
\phi_{i1} = [x_1 \varphi_{11}], \quad \phi_{ik} = \begin{bmatrix}
(-1)^{k-1}\varphi_{ik-1} & 0 \\
x_1I & \varphi_{ik}
\end{bmatrix}
$$

for $k = 2, 3, \ldots, g - 1$,

$$
\phi_{ig} = \begin{bmatrix}
\varphi_{ig-1} \\
-x_i
\end{bmatrix},
$$

(4.1)

Proof. Clear. □

Lemma 4.2. With the notation as above, let $t = \left(\begin{smallmatrix} g \\ 2 \end{smallmatrix}\right)$. Then, for each $i$

1. Every column of $\phi_{i2}$ has exactly two nonzero entries.
2. The number of nonzero rows in each $g \times 2$ submatrix of $\phi_{i2}$ is greater than 2, that is, 3 or 4.

Proof. This follows from the matrix form of $\phi_{i2}$ (see (2.1) and (2.2)). □

Now we can describe the special properties of $\phi_{i2}$ in (4.1).

Proposition 4.3. With the notation as above and hypotheses:

1. $\phi_{i2}$ has $g$ disjoint pairs $(S_k, T_k)$ of a $g \times (g - 1)$ submatrix $S_k$ and a $g \times (t - g + 1)$ submatrix $T_k$;
2. By removing the $i$th row and interchanging columns of $\phi_{i2}$, each pair $(S_k, T_k)$ can be reduced to a pair $(\bar{S}_k, \bar{T}_k)$, where $\bar{S}_k$ is a $(g - 1) \times (g - 1)$ diagonal matrix whose determinant is $x_1^{g-1}$, up to sign, and $\bar{T}_k$ is the second differential map in the Koszul complex $K(y_k)$.

Proof. (1) The first statement follows from the second statement.

(2) It is enough to prove the case $i = 1$. For the sake of simplicity, $\phi_{12}$ can be written as the form

$$
\phi_{12} = \begin{bmatrix}
-\varphi_{11} & 0 \\
x_1I & \varphi_{12}
\end{bmatrix}
$$

(4.2)

Let $S_1 = \phi_{12}(1, 2, \ldots, g | 1, 2, \ldots, g - 1)$ and $T_1 = \phi_{12}(1, 2, \ldots, g | g, g + 1, \ldots, t)$. Then clearly, $S_1$ and $T_1$ are disjoint. Taking $S_1 = x_1I$ and $T_1 = \varphi_{12}$ as submatrices of $\phi_{12}$, it is clear that $\det S_1 = (x_1)^{g-1}$ and $T_1$ is the second differential map in the Koszul complex $K(y_1)$. Let $k > 1$ be an integer with $2 \leq k \leq g$. It follows from Lemma 4.2 that every row of $\phi_{12}$ consists of exactly $g - 1$ nonzero entries and exactly $t - g + 1$ zero entries. Choose $S_k$ to be a $g \times (g - 1)$ submatrix of $\phi_{12}$ such that all the entries of the $k$th row are nonzero, and $T_k$ to be a $g \times (t - g + 1)$ submatrix of $\phi_{12}$ such that all the entries of the $k$th row are zero. Then clearly $S_k$ and $T_k$ are disjoint. Let $S'_k$ and $T'_k$ be the submatrices of $S_k$ and $T_k$ obtained by removing the $k$th row of $S_k$ and $T_k$, respectively. By the part (1) of Lemma 4.2, every column of $S'_k$ has exactly one
nonzero entry. We observe from (4.2) that the nonzero entry in the \(l\)th column of \(S'_k\) is either \(x_k\) or \(-x_k\) for \(l = 1, 2, \ldots, g-1\). The part (2) of Lemma 4.2 implies that every row of \(S'_k\) has exactly one nonzero entry. This implies that interchanging columns of \(S'_k\) produces a \((g-1) \times (g-1)\) diagonal matrix \(\tilde{S}_k\) whose main diagonal entries are either \(x_k\) or \(-x_k\). It follows from the construction of \(T'_k\) and Lemma 4.2 that every column of \(T'_k\) has exactly two nonzero entries and the number of nonzero rows in each \((g-1)\times2\) submatrix of \(T'_k\) is 3. Since \(T'_k\) has \(t - g + 1 = \binom{g-1}{2}\) columns and \(g-1\) rows, interchanging columns of \(T'_k\) (if necessary) gives us the second differential map \(\tilde{T}_k\) in the Koszul complex \(K(y_k)\) (see (4.2)). Actually, \(\tilde{T}_k\) has the form

\[
\tilde{T}_k = \begin{bmatrix}
h_k & 0 \\
d_1 & h'_k
\end{bmatrix},
\]

where

\[
h_k = \begin{bmatrix}
-x_2 & -x_3 & \cdots & -\hat{x}_k & \cdots & -x_g
\end{bmatrix},
\]

\[
d_1 = \text{diag}(x_1, x_1, \ldots, x_1),
\]

\[
h'_k = \text{the second differential map in the Koszul complex } K(y_{1k}) \text{ for } y_{1k} = x_2, x_3, \ldots, \hat{x}_k, \ldots, x_g.
\]

Thus we have the desired one \(\tilde{T}_k\). \(\square\)

To define the ideal \(K_{g-1}(\phi_{12})\) associated with the map \(\phi_{12}\) we need further properties of \(\phi_{12}\).

**Theorem 4.4.** (1) With the notation as in Proposition 4.3, for each \(1 \leq k \leq g\), a pair \((\tilde{S}_k, \tilde{T}_k)\) of matrices given in Proposition 4.3 is uniquely determined.

(2) If for each \(k\), \(K_{g-2}(\tilde{T}_k)\) is the ideal generated by the elements \(x_1, x_2, \ldots, \hat{x}_k, \ldots, x_g\) given in the proof of Proposition 4.3, then \(K_{g-2}(\tilde{T}_k)\) has grade \(g-1\).

**Proof.** (1) This follows from Lemma 4.2.

(2) The second part is also clear since \(x_1, x_2, \ldots, \hat{x}_k, \ldots, x_g\) is a regular sequence on \(R\). \(\square\)

Thus Theorem 4.4 enables us to define a complete matrix of grade \(g\). With an induction argument, we may call \(\tilde{T}_k\) given in Theorem 4.4 the complete matrix of grade \(g-1\) in the following sense.

**Definition 4.5.** Let \(R\) be a commutative ring with identity. Let \(g > 3\) and \(t = \binom{g}{2}\) be integers. A \(g \times t\) matrix \(f = (f_{ij})\) over \(R\) is said to be complete of grade \(g\) if

(1) \(f\) has \(g\) disjoint pairs \((S, T)\) of a \(g \times (g-1)\) submatrix \(S\) and a \(g \times (t-g+1)\) submatrix \(T\);

(2) By removing a row and interchanging columns, each pair \((S, T)\) can be reduced to a pair \((\hat{S}, \hat{T})\), where \(\hat{S}\) is a \((g-1) \times (g-1)\) diagonal matrix with
det(\vec{S}) = x^{g-1} \text{ for some } x \in R, \text{ and } \vec{T} \text{ is the complete matrix of grade } g - 1 \text{ with grade } \mathcal{K}_{g-2}(\vec{T}) = g - 1.

The following example illustrates Definition 4.5.

Example 4.6. Let \( x, y, z, u, w \) be a regular sequence in a Noetherian local ring \( R \). Let

\[
\begin{bmatrix}
y & z & u & w & 0 & 0 & 0 & 0 & 0 \\
-x & 0 & 0 & 0 & z & u & w & 0 & 0 \\
0 & -x & 0 & 0 & -y & 0 & 0 & u & w \\
0 & 0 & -x & 0 & 0 & -y & 0 & -z & w \\
0 & 0 & 0 & -x & 0 & 0 & -y & 0 & -z & -u 
\end{bmatrix}
\]

The similar argument as in Example 3.4 shows that \( f \) satisfies the properties in Proposition 4.3 and the part (2) of Theorem 4.4.

The following theorem is an easy generalization of Theorem 3.5.

Theorem 4.7. Let \( g > 3 \) and \( t = \binom{g}{2} \) be integers. A \( g \times t \) matrix \( f = (f_{ij}) \) over \( R \) is a complete matrix of grade \( g \).

1. Every column of \( f \) has exactly two nonzero entries.
2. The number of nonzero rows in each \( g \times 2 \) submatrix of \( f \) is greater than 2.
3. Each pair \((\vec{S}, \vec{T})\) of matrices given in Definition 4.5 is uniquely determined.

Proof. The proofs are essentially similar with those of Theorem 3.5. □

Now we define an ideal \( \mathcal{K}_{g-1}(f) \) generated by the entries in the \( (g-1) \times (g-1) \) matrices \( \vec{S} \) derived from a given complete matrix \( f \) of grade \( g \) in Theorem 4.7.

Definition 4.8. Let \( g > 3 \) and \( t = \binom{g}{2} \) be integers. Let \( f \) be a \( g \times t \) complete matrix of grade \( g \). For \( i = 1, 2, \ldots, g \), we let \( \vec{S}_i \) be a unique \((g-1) \times (g-1)\) diagonal matrix extracted from \( f \) in the part (3) of Theorem 4.7 such that \( \det(\vec{S}_i) = x_i^{g-1} \) is nonzero for some \( x_i \in R \). We define \( \mathcal{K}_{g-1}(f) \) to be the ideal generated by the \( x_i \)'s, that is,

\[
\mathcal{K}_{g-1}(f) = (x_1, x_2, \ldots, x_g).
\]

Let \( f = (f_{ij}) \) be a \( g \times t \) complete matrix of grade \( g \). It follows from the properties (1) and (2) of Theorem 4.7 that interchanging columns of \( f \) transforms \( f \) to the following form.

\[
(4.3) \quad f = \begin{bmatrix} h_1 & 0 \\ d_1 & h_2 \end{bmatrix},
\]

where

\[
\begin{align*}
h_1 &= \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1g-1} \end{bmatrix}, \\
d_1 &= \text{diag}(f_{21}, f_{32}, \ldots, f_{gg-1}), \quad h_2 = \text{a complete matrix of grade } g - 1.
\end{align*}
\]
By applying the method of (3.5) and (3.6) in the case of a complete matrix of grade 4 to the given $f$, we have

$$K_{g'-2}(T_1) = (\hat{f}_{21}, f_{11}, f_{12}, \ldots, f_{1g-1})$$

and

$$K_{g'-2}(T_i) = (f_{21}, f_{11}, f_{12}, \ldots, \hat{f}_{i-1,i-1}, f_{i,i}, \ldots, f_{1g-1}) \quad \text{for } i = 2, 3, \ldots, g,$$

where $\hat{f}_{ii}$ indicates that $f_{ii}$ is to be omitted.

Hence

$$K_{g'-2}(T_i) \subseteq K_{g'-1}(f) = (f_{21}, f_{11}, f_{12}, \ldots, f_{1g-1}) \quad \text{for each } i.$$

The following lemma will be used in proving the structure theorem for complete intersections of grade $g > 3$.

**Lemma 4.9.** Let $\mathbf{x} = x_1, x_2, \ldots, x_g$ be a regular sequence on $R$ and $\mathcal{E}$ a minimal free resolution of $R/(\mathbf{x})$. If $\varphi_2$ is the second differential map of $\mathcal{E}$, then $\varphi_2$ is a complete matrix of grade $g$.

**Proof.** Let $\mathcal{K}(\mathbf{x})$ be the Koszul complex defined by the regular sequence $\mathbf{x} = x_1, x_2, \ldots, x_g$ and $d_2$ the second differential map in $\mathcal{K}(\mathbf{x})$. We have shown in Proposition 4.3 and the part (2) of Theorem 4.4 that $d_2$ is a complete matrix of grade $g$. Let $F$ be the free $R$-module with the ordered basis $\{e_1 < e_2 < \cdots < e_g\}$. Then $\wedge^2 F$ is a free $R$-module with the ordered basis $\{e_1 \wedge e_2 < e_1 \wedge e_3 < \cdots < e_{g-1} \wedge e_g\}$. Let $t = \binom{g}{2}$ be an integer. Let

$$\mathcal{E} : 0 \longrightarrow F_g \xrightarrow{\varphi_g} F_{g-1} \xrightarrow{\varphi_{g-1}} \cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} R$$

be the minimal free resolution of $R/(\mathbf{x})$ such that $F_1$ and $F_2$ are free $R$-modules with the ordered bases $\{v_1 < v_2 < \cdots < v_g\}$ and $\{w_1 < w_2 < \cdots < w_1\}$, respectively. Then we have a commutative diagram

$$\begin{array}{ccc}
\wedge^2 F & \xrightarrow{d_2} & \wedge^1 F \\
\downarrow{\psi_2} & \circlearrowright & \downarrow{\psi_1} \\
F_2 & \xrightarrow{\varphi_2} & F_1
\end{array}$$

where $\psi_1$ and $\psi_2$ are order preserving isomorphisms as free $R$-modules. Since $\psi_1(e_k) = v_k$ for $k = 1, 2, \ldots, g$ and $\psi_2$ maps the $i$th basis element in $\wedge^2 F$ to the $i$th basis element $w_i$ in $F_2$ for $i = 1, 2, \ldots, t$, the commutativity implies that $d_2$ and $\varphi_2$ have the same matrix representation. Thus $\varphi_2$ is a complete matrix of grade $g$ since $d_2$ is a complete matrix of grade $g$. \hfill \square

Now we can describe a structure theorem for complete intersections of grade $g > 3$.

**Theorem 4.10.** Let $R$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$.

1. Let $g > 3$ be an integer and $t = \binom{g}{2}$. Let $F$ and $G$ be free $R$-modules with rank $F = g$ and rank $G = t$. Let $f = (f_{ij}) : G \rightarrow F$ be a complete matrix of
grade $g$ whose image is contained in $mF$. With the notation as in Theorem 4.7, we assume that $K_{g-2}(T_1) + K_{g-2}(T_2)$ has grade $g$ for some $i,j(1 \leq i \neq j \leq g)$. Then the ideal $K_{g-1}(f)$ is a complete intersection of grade $g$. 

(2) Let $I = (x_1, x_2, \ldots, x_g)$ be a complete intersection of grade $g$ and let \( \varphi_2 \) be the minimal free resolution of $R/I$. Then $\varphi_2$ and the transpose of $\varphi_{g-1}$ satisfy (1).

Proof. (1) We showed in Theorem 3.7 that the first part of the theorem is true for the case of $g = 4$. Let $f = (f_{ij})$ be a $g \times t$ complete matrix of grade $g$. As shown in Proposition 4.3, interchanging columns of $f$ transforms $f$ to the form of (4.3). So we may assume that $f$ has the form of (4.3). Then we have

\[
K_{g-1}(f) = \left( f_{21}, f_{11}, f_{12}, \ldots, f_{1g-1} \right).
\]

Since $K_{g-2}(T_1) + K_{g-2}(T_2)$ has grade $g$ for some $i,j(i \neq j)$, it follows from (4.4) and (4.5) that $K_{g-1}(f)$ is a complete intersection of grade $g$. Let \( x = f_{21}, f_{11}, f_{12}, \ldots, f_{1g-1} \). Then \( y_1 = f_{21}, f_{11}, f_{12}, \ldots, f_{1g-1} \) and each \( y_i = f_{21}, f_{11}, f_{12}, \ldots, f_{i-11}, \ldots, f_{1g-1} \) for $i > 1$ are regular sequences. From (4.4), $f_{21}$ is regular on $R/K_{g-2}(T_1)$, and $f_{1i-1}$ is regular on $R/K_{g-2}(T_1)$ for $i > 1$. Let $G_i$ be a complex of free $R$-modules such that

\[
G_1 : 0 \longrightarrow R \xrightarrow{f_{21}} R, \\
G_i : 0 \longrightarrow R \xrightarrow{f_{1i-1}} R.
\]

Then by the part (1) of Lemma 4.1, $G_i \otimes \mathbb{K}(y_i)$ is a minimal free resolution of $R/K_{g-1}(f)$.

(2) We showed in Theorem 3.7 that the part (2) holds for the case of $g = 4$. Let $I = (x_1, x_2, \ldots, x_g)$ be a complete intersection of grade $g$ and $I' = (x_2, x_3, \ldots, x_g)$ be a complete intersection of grade $g-1$. The same argument as in the proof of the part (2) of Theorem 3.7 says that $\varphi_2$ in (4.6) is of the form

\[
\varphi_2 = \begin{bmatrix}
\tilde{\varphi}_1 & 0 \\
\tilde{d} & \tilde{\varphi}_2
\end{bmatrix},
\]

where

\[
\tilde{\varphi}_1 = \begin{bmatrix}
-x_2 & -x_3 & \cdots & -x_g
\end{bmatrix}, \quad \tilde{d} = \text{diag}(x_1, x_1, \ldots, x_1),
\]

and $\tilde{\varphi}_2$ is the second differential map of the minimal free resolution of $R/I'$.

Lemma 4.9 says that $\tilde{\varphi}_2$ is a complete matrix of grade $g-1$. Since $x_1, x_2, \ldots, x_g$ is a regular sequence on $R$, Lemma 4.9 implies that $\varphi_2$ is a complete matrix of grade $g$. We observe that every row of $\varphi_2$ consists of $g-1$ nonzero entries and
$t - g + 1$ zero entries. The similar argument as in the proof of Proposition 4.3 gives us the following: Let $(S_i, T_i)$ be a pair of a $(g - 1) \times (g - 1)$ diagonal matrix and a $(g - 1) \times (t - g + 1)$ complete matrix of grade $g - 1$. Then for $i = 1, 2, \ldots, g$,
\[
\det S_i = \pm x_i^{g-1}, \quad K_{g-2}(T_i) = (x_1, x_2, \ldots, x_{i+1}, \ldots, x_g).
\]
So we have
\[
K_{g-1}(\varphi_2) = (x_1, x_2, \ldots, x_g), \quad \text{and} \quad K_{g-2}(T_i) + K_{g-2}(T_j) = K_{g-1}(\varphi_2)
\]
for some $i \neq j$.

We know that each $K_{g-2}(T_i)$ has grade $g - 1$, and $K_{g-1}(\varphi_2)$ is a complete intersection of grade $g$. Hence $\varphi_2$ satisfies the part (1) of Theorem 4.10. Since every complete intersection is Gorenstein, $\mathbb{F} \cong \mathbb{F}^*$ as complexes. So $\mathbb{F}^*$ is the minimal free resolution of $R/I$. The same argument as in the proof of the part (2) of Theorem 3.7 for $\mathbb{K}(x)$ and $\mathbb{F}^*$ gives us the proof that the transpose of $\varphi_{g-1}$ is a complete matrix of grade $g$.

It should be noticed that Theorem 3.7 is just the special case of Theorem 4.10. The following example illustrates how Theorem 4.10 works.

**Example 4.11.** Let $\mathbb{C}$ be the field of the complex numbers and $R$ the formal power series ring $\mathbb{C}[[x_{ij}, y, z, w, u | 1 \leq i, j \leq 3]]$ over $\mathbb{C}$ with indeterminates $x_{ij}, y, z, w, u$. Consider a $3 \times 3$ matrix $X$ and a $3 \times 3$ alternating matrix $Y$
\[
X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & w & z \\ -w & 0 & y \\ -z & -y & 0 \end{bmatrix}.
\]
Define
\[
Z_1 = \sum_{i=1}^{3} Y_i x_{i1}, \quad Z_2 = \sum_{i=1}^{3} Y_i x_{i2}, \quad Z_3 = \sum_{i=1}^{3} Y_i x_{i3}, \quad v = \det X.
\]
Then $I = (Z_1, Z_2, Z_3, v)$ is an almost complete intersection of grade 3 of type 3 [1, 5]. Assume that $x = Z_1, Z_2, Z_3$ is a regular sequence on $R$. Then
\[
J = (z) : I = (Y_1, Y_2, Y_3) = (y, z, w).
\]
Since $v$ is not contained in the ideal $J, I \cap J = (z)$. Hence $I$ is geometrically linked to $J$ by a regular sequence $z$. Thus by Theorem 2.3, $K = I + J = (y, z, w, v)$ is a complete intersection of grade 4. So $x = y, z, w, v$ is a regular sequence on $R$. We may assume that $u$ is a regular element on $R/K$. Thus $H = (y, z, w, v, u)$ is a complete intersection of grade 5. Let
\[
\mathbb{K}(u) : 0 \longrightarrow R \xrightarrow{u} R
\]
be a complex of free $R$-modules and $R$-maps. Then $\mathbb{H} = \mathbb{K}(u) \otimes \mathbb{K}(x)$ described as in the part (2) of Lemma 4.1 is the minimal free resolution of $R/H$. Let $\varphi_2$ be the second differential map in $\mathbb{H}$. Since $y, z, w, v, u$ is a regular sequence
on \( R \), by Lemma 4.9, \( \phi_2 \) is a complete matrix of grade 5. It is easy to show that \( K_3(\phi_2) = (u, y, z, w, v) \) is a complete intersection of grade 5. Moreover, we let \( \bar{T}_i \) be a \( 4 \times 6 \) complete matrix of grade 4 with the same notation, \( \bar{T}_i \) in Definition 4.5. Then we have

\[
K_3(\bar{T}_i) = (y, z, w, v), \quad K_3(\bar{T}_2) = (u, z, w, v), \quad K_3(\bar{T}_3) = (u, y, w, v),
\]

\[
K_3(\bar{T}_4) = (u, y, z, v), \quad K_3(\bar{T}_5) = (u, y, z, w).
\]

Hence \( K_3(\bar{T}_i) + K_3(\bar{T}_j) = K_4(\phi_2) \) for some \( i \neq j \). This illustrates the Theorem 4.10.

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